RECENT DEVELOPMENTS IN REGGEON FIELD THEORY

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Abstract:

In the last few years Reggeon Field Theory (RFT) has been developed into a very powerful tool for analyzing the complex angular momentum structure of high energy scattering amplitudes. In a previous review of this subject (Abarbanel et al. [8]) RFT was motivated, formulated, and its early results were discussed. These issues will be mentioned only briefly in the introduction to the present review in order to make it self-contained. It will then be followed by a discussion of the recent developments and progress in the field. A major part of this review is devoted to the discussion of various efforts made to elucidate the \(s\)-channel content of Reggeon Field Theory. The formulation of RFT on a lattice and the analysis of production amplitudes are emphasized as the main approaches towards understanding the implication of RFT on \(s\)-channel unitarity issues. The path integral and the Hamiltonian formulations of RFT on a lattice, along with their results, are treated in detail. The work done on production amplitudes, in particular cut Reggeon field theory, its derivation and applications, are presented. The problem of the approach to scaling, the evaluation of the transition energy and scaling functions are discussed as they emerge from the representations of Reggeon Green's functions. Practical problems such as the confrontation of the RFT results with the experimental data and the relevance of different approximations are also analyzed.

1. Introduction

1.1. Motivation and formulation of Reggeon Field Theory—brief summary

It has been more than sixteen years since Regge poles were successfully introduced into elementary particle physics (Regge [152], Chew and Frautschi [66]). The \(t\)-channel viewpoint of high energy scattering seems today even more appealing at the large accelerator energies, where the many body final states result in a complicated \(s\)-channel picture. Throughout these years, however, it has been realized that the initial picture of simple Regge pole trajectories, although adequate for a rough description of the data, is unacceptable either phenomenologically or theoretically. Hadronic scattering amplitudes seem to have richer complex angular momentum structure and it is difficult and sometimes impossible with Regge poles alone, to account for many experimental results such as:

(i) Polarization in \(\pi N\) charge exchange (fig. 1),
(ii) The logarithmic rise of the total cross sections (fig. 2),
(iii) Absorption and rescattering effects in elastic and inelastic scattering (e.g., the dip structure in the elastic cross section—fig. 3),
(iv) Target polarization dependence of the inclusive cross section in the \(x_F \rightarrow 1\) region (fig. 4),
(v) Diffraction dissociation into high mass and long range correlation in inclusive reactions (fig. 5).

These are only a few out of many experimental results that point to the existence of Regge cuts contributions in hadronic amplitudes.

From a theoretical point of view, Regge cuts are indeed necessary. Amati et al. [17], Mandelstam
I and Polkinghorne [151] have shown that a relativistic $S$ matrix, which has poles in the complex angular momentum plane $j$, will generate branch point cuts in $j$ as well. In Regge terms, the branch point singularities can be viewed as resulting from the exchange of several Regge poles (fig. 6b). Imposing unitarity in the $t$ channel gives rise to additional difficulties if the leading singularity is a simple Regge pole, essentially because of the non-linearity of the unitarity relations [105]. Also, from the point of view of $s$-channel unitarity, an isolated leading pole, with intercept at one, causes very severe difficulties. Such a leading pole singularity leads to violation of the Froissart bound (Finkelstein–Kajantie “disease” [90]). It was shown that a leading pole $P$ with $\alpha(0) = 1$ implies the vanishing of the triple $P$ coupling at $t = 0$ when all momentum transfer are zero [1, 77]. Furthermore, through a sequence of inequalities it can be shown that $P$ finally decouples from elastic processes and therefore does not contribute to $\sigma_t$ [37, 122].

One has to move further away from the single Regge poles description of hadronic amplitudes (fig. 6a), through a Regge poles + Regge cuts picture (fig. 6b), to a more complicated picture (fig. 6c), where Reggeons also “interact”. The simplest Reggeon interaction diagram is shown in fig. 7. It is a phenomenological description of diffraction dissociation into high mass. The sum on the undetected particles is described as a particle-Pomeron total cross section dominated by a Regge pole. The interaction between Pomerons is very appealing also from a theoretical point of view. The enhancement of the two Pomeron branch cut (fig. 8) at $\alpha(t) = 2(\alpha(t/4) - 1) + 1$ by the pole at $\alpha(0) = 1$, as shown in fig. 9, was discussed by Gribov [109]. It gives rise to a triple Pomeron interaction of strength, $r_0$, in

![Fig. 2. Total cross sections from $s = 10$ to $=3600$ GeV².](image)
an enhanced two Pomeron cut (fig. 9), and a double enhanced cut (fig. 10). In principle, many other forms of Reggeon interactions are also present.

The phenomenological and theoretical arguments briefly mentioned above support a picture of a rich complex angular momentum structure of hadronic scattering amplitudes. Viewed in fig. 6c, the scattering of hadrons A and B is associated with the "emission" of Reggeons by A; they then "propagate" and "interact" until "absorbed" by particle B. It has been realized by Gribov [109] that...
Recent developments in Reggeon field theory

Fig. 6. (a) Regge pole, (b) Regge cut, (c) Regge cut including triple-Reggeon interactions.

Fig. 7. Diffraction dissociation into high mass and the triple Pomeron interaction in a Mueller-Regge diagram.

such a picture of "propagation" of Reggeons will be adequately described by field theory techniques. One of the goals of such a field theory is to provide us with a set of rules for the calculation of the partial-wave amplitude, $F(j, t)$,

$$ F(j, t) = \int ds s^{-1} \text{Im} \, T(s, t). $$

(1.1)

$T(s, t)$ is the elastic scattering amplitude and its relation to $F(j, t)$ is given by (note $\tau = +1$)

$$ T(s, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\xi F(j, t) s^\xi. $$

(1.2)

$\xi$ is the signature corresponding to $T(s, t)$.

Reggeon unitarity relations [108, 167, 2] which will be discussed below serve as a basis and as an indication of the usefulness of Reggeon field theory (RFT) for the study of branch cuts in the $j$ plane. Figure 11 represents the two Reggeon cut obtained from the exchange of two Reggeons in the $t$ channel and is given by:

$$ F_2(j, q^2 = -t) = \int d^2q_1 d^2q_2 \delta^2(q_1 + q_2 - q) \frac{N_2^A(j, q_1)N_2^B(j, q_1)}{E E_1 E_2}. $$

(1.3)

$F_2(j, t)$ has a branch point in $j$ at $\alpha^{(2)}(t) = 1 + 2 (\alpha(t/4) - 1)$, and we denoted $1 - j$ by $E$. $N_2^A$ and $N_2^B$ describe the coupling of the two Reggeons to particles A and B, respectively. The discontinuity across the cut in $E$ is:

$$ \text{Disc}_{E} F_2(E, q^2) = \int d^2q_1 d^2q_2 \delta^2(q_1 + q_2 - q) \delta(E - E_1 - E_2) N_2^A(E^+, q_1)N_2^B(E^-, q_1). $$

(1.4)

$N_2^A$ and $N_2^B$ are evaluated above and below the two Reggeon cut, respectively. Similarly, one can think of taking the discontinuity across the $n$ Reggeon cut of fig. 6, which will also result in the replacement of a denominator by a delta function. Reggeon-Reggeon "scattering" has already been encountered above (e.g., fig. 6c) and it will be useful to introduce in the following discussion Reggeon

Figs. 8, 9, 10. The two-Pomeron cut, enhanced two-Pomeron cut, and double enhanced two-Pomeron cut, respectively.
n point Green's functions. For example, $M_4(E_k)$ in fig. 12 represents a two Reggeon scattering amplitude; its discontinuity across the $n$ Reggeon cut is given by:

$$\text{Disc.}_EM_4(E_k) = \int \prod_{i=1}^n d^2q_i \delta^2 \left( k_i + k_2 - \sum q_i \right) \delta \left( E_1 + E_2 - \sum E_i \right)$$

$$\times M_{2,n}(E^+, E_1, k_1, k_2, E_n, q_i) M_{2,n}(E^+, E_1, E_3, k_3, k_4, E_n, E_1).$$

(1.5)

There is a striking similarity between eq. (1.5) and more familiar unitarity relations for the scattering process of two particles with initial and final energies and momentum $E_i, k_i$ ($j = 1$–4). Indeed, the Reggeon appears in eq. (1.5) as a quasiparticle living in a two-dimensional space with a dispersion relation $E(k^2) = 1 - \alpha(k^2)$ (eq. (1.3)). In case of a linear trajectory we have:

$$E(k^2) = 1 - \alpha(k^2) = \alpha'k^2 - (1 - \alpha(0)) = \alpha'k^2 - \Delta_0$$

(1.6)

and a simple Regge pole (e.g., fig. 6a) given by:

$$\frac{-i}{j - \alpha(k^2)} = \frac{i}{1 - j - \alpha'k^2 - (1 - \alpha(0))} = \frac{i}{E - \alpha'k^2 - \Delta_0} = G^{1,1}(E, k)$$

(1.7)

will be interpreted now as the “bare propagator” of a “free” Reggeon. The analogy can be further proceeded with the observation that for an intercept $\alpha(0) = 1$, the location of all the branch points [see eq. (1.3) for the case $n = 2$] will be at

$$j_{B.P.} = \alpha^n(t)|_{t=0} = 1 + n(\alpha(t/n^2) - 1)|_{t=0} = 1$$

(1.8)

and are independent of $n$. In $E$ space (fig. 13), all $n$ “particles” thresholds collide as they should for a massless particle theory. We see, therefore, that in order to calculate the multi-Reggeon exchange contribution to high energy scattering we have to analyze the infrared behavior of an interacting massless theory.

Reggeon unitarity relations (e.g., eq. (1.5)), on which the analogous field theory for the Reggeon is based, were derived and studied extensively in several different frameworks and therefore are considered to be correct in general: (i) Reggeon unitarity relations were derived from the analysis of hybrid Feynman graphs of the type of fig. 14 at high $s$ and small $t$ [109] without using specific field theory representation of the Reggeon, (ii) These relations were also derived in the framework of the dual resonance model [133], (iii) Reggeon unitarity relations are common in a wide class of multiperipheral models [2], (iv) Finally, they were also derived from multiparticle $t$-channel unitarity,
and the following important point was shown to be correct: A theory that satisfies all Reggeon unitarity relations also satisfies t-channel unitarity [108, 167]. This last point should be emphasized in particular since a field theory for Reggeons satisfies, by construction, all Reggeon unitarity equations, and is therefore a t-channel unitary theory. This is certainly one of the major virtues of RFT. A presentation of the physical basis and a detailed discussion on the derivation of Reggeon calculus can be found in the review of Baker and Ter-Martirosyan [24].

The idea behind a formulation of a field theory for the Reggeon is to treat the Reggeon, which is normally considered to be a composite object of bound states of the underlying dynamics, as a quasiparticle. One hopes then to study the contributions of Regge cuts to the complex angular momentum structure of hadronic scattering amplitudes, without a complete knowledge of the underlying dynamics. The Reggeon is treated as a quasiparticle in the same way one treats collective excitations in solid state physics. The quasiparticle to be described by the field theory is the Reggeon of energy, $E(1-j)$, and two dimensional momentum $k$. The conjugate variable to $E$ is the “time” $\tau = -iY = -i \log s$ (see eq. (1.1)), and the conjugate variable to $k$ is $x = b$, the impact parameter. The limit $\log s \to \infty$ corresponds to $E \to 0$, and $t = -k^2 \to 0$ corresponds to $x^2 \to \infty$. The field $\psi(x, \tau)$, which is associated with the Reggeon, satisfies an equation of motion compatible with the energy momentum relation in eq. (1.6); it is therefore derived from the Lagrangian density:

$$\mathcal{L}_{\text{free}}(x, \tau) = \frac{1}{2} \bar{\psi} \frac{\partial^2}{\partial \tau^2} \psi(x, \tau) - \alpha_0' \nabla \bar{\psi} \cdot \nabla \psi - \Delta_0 \bar{\psi} \psi. \tag{1.9}$$

The equation of motion for $\psi$ and $\bar{\psi}$ is:

$$i \frac{1}{\partial \tau} \psi(x, \tau) = -\alpha_0' \nabla^2 \psi + \Delta_0 \psi. \tag{1.10}$$

Equations (1.9) and (1.10) reproduce, of course, eqs. (1.6) and (1.7).

It was argued above that a complete t-channel description of an hadronic scattering amplitude should include also Reggeon interactions. In particular, in a field theory for the Pomeron, which will be discussed later, the triple Pomeron interaction mentioned above will be included in the Lagrangian. It is a non hermitian term which assures a relative negative sign between the two-Pomeron cut and the single Pomeron exchange (Gribov [109]):

$$\mathcal{L}_{\text{int.}} = -i \frac{1}{2} r_0 \bar{\psi}(\bar{\psi} + \psi) \psi. \tag{1.11}$$

Note that the non hermitian interaction implies that $\bar{\psi}(x, \tau)$ is not the hermitian conjugate of $\psi(x, \tau)$. One can choose however one particular $y$, say $y = 0$, at which $\bar{\psi}(x, 0) = \psi^\dagger(x, 0)$. At a later time the fields evolve as dictated by the Heisenberg equations of motion, whose non hermitian Hamiltonian will imply $\bar{\psi}(x, \tau) \neq \psi^\dagger(x, \tau)$ at $\tau \neq 0$ [41]. Certain symmetry properties of the theory characterizing the

Fig. 14. The description of the two-Reggeon cut graphs in terms of hybrid Feynman diagrams.
relation between \( \tilde{\psi} \) and \( \psi^\dagger \), will be one of the main concerns in the following sections (see in particular section 7).

Clearly, in addition to eq. (1.11) there are other possible derivative and non-derivative Pomeron interactions that can be added to \( \mathcal{L}_{\text{int}} \), and this important issue had been discussed in several papers (Migdal et al. [139], Abarbanel andBronzan [5], Brower and Ellis [38], Calucci and Jengo [48], Bardeen et al. [22]). It was shown that the infrared (\( E = 1 - j \rightarrow 0 \)) behavior of the theory is not affected if other Pomeron self interaction terms are added in eq. (1.11); they are often called "infrared irrelevant terms" [126, 134]. This is a common feature in theories possessing a fixed point (to be discussed below); the asymptotic behavior is determined by the value of the parameters at the fixed point rather than by their physical value. The very high energy behavior of the theory is independent of the details of the Pomeron's interactions. This is one of the most appealing features of Reggeon field theory.

The set of the coupled discontinuity equations (Reggeon unitarity relations) of the type of eq. (1.4) is automatically satisfied in the interacting field theory for the Pomeron formulated above where one has to calculate the \( n \) incoming \( m \) outgoing Reggeon's Green's functions (fig. 15a),

\[
G^{n,m}(x_1 \tau_1 \ldots x_n \tau_n, x_{n+1} \tau_{n+1} \ldots x_{n+m} \tau_{n+m}) = \langle 0 | T[\tilde{\psi}(x_1 \tau_1) \ldots \tilde{\psi}(x_n \tau_n)\psi(x_{n+1} \tau_{n+1}) \ldots \psi(x_{n+m}, \tau_{n+m})]|0 \rangle.
\] (1.12)

A perturbation theory calculation in powers of \( r_0^2 \) of \( G^{n,m} \), using the Lagrangian in (1.9) and (1.11) is, however, not the appropriate technique for the analysis of the high energy behavior. As seen above, if the leading singularity is at \( \alpha(0) = 1 \ (E(0) = 0) \), there will be an accumulation of an infinite number of significantly important cuts at small \( t \) and \( E \); therefore, many (perhaps infinite) orders in the perturbation series had to be summed in order to find the correct high energy limit. In the angular momentum plane the \( E \rightarrow 0 \) limit of \( G^{n,m}(k_i E_i) \) is of interest. The presentation and discussion of the techniques and tools which were developed for a nonperturbative approach in the analysis of the infrared limit of the theory is the main goal of this report.

Much of the work done in RFT has benefited from the intuition one gets from the analogy of the infrared problem encountered here with the similar problem found in second order phase transitions in statistical mechanics. There are several ways to state this analogy, which although differing in details, they share a common viewpoint. Due to Pomeron interactions the mass gap (eq. (1.9)) \( \Delta_0 = 1 - \alpha_0 \) is renormalized to a different value \( \Delta \). The requirement that the renormalized gap will vanish, namely

\[
G^{R}_{R} = \frac{\Delta_0 - \Delta}{\Delta_0} G^{R}_{R}.
\]

Fig. 15. (a) \( n \) incoming \( m \) outgoing Pomeron Green's function, (b) Pomeron's renormalized propagator, (c) Renormalized Pomeron's three-point Green's function.
that the renormalized Pomeron intercept, \( \alpha(0) \), will be equal to one, is implemented if \( \alpha_0 \) retains a specific value, \( \alpha_{0c} \), called the critical value. Only at that value of \( \alpha_0 \) we are dealing with a zero mass (gap) theory, and the forward elastic scattering amplitude does not show any more a power behavior \( s^{-\Delta} \). The zero mass limit in solid state is reached in an analogous way. Close to their critical point as \( T \to T_c \), solids show the appearance of long range correlations; the order parameter becomes very large compared to the size of the unit cell; the effective mass in the theory goes to zero and gives rise to long range forces in the system. At that point, the "critical behavior" of the solid is determined not by its local short range forces, but by the overall symmetry and dimensionality of the crystal; it shows a so-called "universal behavior".

The field theory approach to critical phenomena can be demonstrated, for example, in a \( \lambda \phi^4 \) theory (see Brezin, LeGuillou and Zinn–Justin [33]). Near the critical point the renormalized mass, \( m(T) \), approaches zero. As the unrenormalized mass, \( m_0 \), reaches a special value, \( m_{0c} \) at \( T = T_c \) (the critical value \( m_0(T = T_c) = m_{0c} \)), then the renormalized mass satisfies \( m(T = T_c) = 0 \), and the system goes through a phase transition.

The successful application of renormalization group technique, commonly used in critical phenomena, attracted new attention to Reggeon calculus and revealed its strength and beauty (Mirgal, Polyakov and Ter-Martirosyan [139], Abarbanel and Bronzan [4]). In solid state physics, as one approaches the critical point, it has been found extremely useful to transform the coordinates with which the system is described (e.g., the spins at lattice sites) into different coordinates which describe averages of larger pieces of the solid (i.e., block spins made of a large number of unit cells – Stanley [155], Kogut and Wilson [126], Ma [134]). In this way the local short range interactions are smoothed out, leaving, however, the important symmetry and dimensionality of the system, unchanged. Such a block size transformation should therefore be a symmetry operation of the effective Hamiltonian describing the system; it is called a renormalization group transformation. As \( T \to T_c \) and the critical point is approached, the description of the critical behavior of the system does not depend on the block size; namely, the parameters of the Hamiltonian near the phase transition are unchanged by the transformation of the block size. We then say that a "fixed point" was reached and the universal behavior of the system takes place; the system goes through a phase transition.

The quantity usually playing the role of temperature in Reggeon field theory is the intercept of the bare Pomeron, \( \alpha_0 \). (Later we will meet in section 7 other parameters that may play this role.) The phase transition which will be discussed here occurs when \( \Delta = 1 - \alpha(0) = 0 \), namely, the renormalized mass gap vanishes and this happens if \( \alpha_0 = \alpha_{0c} \). At this point, which is analogous to \( T = T_c \), the Reggeon Green's function, and therefore the hadronic scattering amplitude, display all the features of a system at its critical point. In particular, its universal scaling behavior reflects the accumulation of the Regge cuts at \( \alpha(0) = 1 \), which is exactly what we wish to understand and calculate. Other interesting problems related to the nature of the ordered phase (\( \alpha_0 > \alpha_{0c} \), which is analogous to \( T < T_c \) region) will be treated in section 7, and the formulation of the theory on the lattice will help in illuminating these issues.

1.2. Renormalization group equation and universal scaling behavior

The renormalization group equation employs the simple observation that unrenormalized quantities obviously do not depend on the location of the renormalization point \( E = -E_N \). In particular for the renormalized single particle irreducible proper vertex function, \( \Gamma^{n,m}_R = Z^{(n+m)/2} \Gamma^{n,m} \) it reads [4]:

\[
\left\{ \xi \frac{\partial}{\partial \xi} - \beta(g) \frac{\partial}{\partial g} + (\alpha' - \xi(\alpha'g)) \frac{\partial}{\partial \alpha'} + \left( \frac{n + m}{2} \gamma(g) - 1 \right) \right\} \Gamma^{n,m}_R(\xi E_i, k_i, g, \alpha', E_N) = 0
\]  

(1.13)
where \( g = (\rho / \alpha ') (E_N / \alpha ')^{D-1} \) is the dimensionless coupling constants. The functions \( \beta, \gamma, \zeta \) are given by:

\[
\beta(g), \gamma(g), \zeta(g') = E_N \left. \frac{\partial}{\partial E_N} (g, \ln Z, \alpha') \right|_{\text{fixed bare parameters}}.
\]

(1.14)

The standard solution to eq. (1.13) (see, for example, the lectures by S. Coleman [69]) is:

\[
\Gamma^{m}_{\text{R}}(E, k, g, \alpha', E_N) = \Gamma^{m}_{\text{R}}(E, k, g, \alpha'(-\ln \xi), E_N) \exp \int_{-\ln \xi}^{0} dt \left[ 1 - \frac{1}{2} (m + n) \gamma(\tilde{g}(t)) \right].
\]

(1.15)

\( \tilde{g} \) and \( \tilde{\alpha}' \) are the effective coupling constant, and the effective slope, respectively, and are solutions of:

\[
\frac{d\tilde{g}(t)}{dt} = -\beta(\tilde{g}(t)), \quad \frac{d\tilde{\alpha}'(t)}{dt} = \tilde{\alpha}'(t) - \zeta(\tilde{\alpha}(t), \tilde{g}(t)), \quad t = \ln \xi.
\]

(1.16)

Since the infrared region, \( (\xi \to 0) \), is of interest here, the behavior of \( g(-\ln \xi) \), which in turn is determined by \( \beta(g) \), is of crucial importance in that region. Although eq. (1.15) is a nonperturbative solution, the only source of information on \( \beta(g) \) comes from its calculation in perturbation theory. First loop calculations, namely, taking into account only the first two terms in figs. 15b and 15c, were carried out by Abarbanel and Bronzan [4], and Migdal et al. [139]. To this order the \( \beta \) function (see fig. 16) is:

\[
\beta(g) = -\left( \frac{4-D}{4} \right) g + A g^3, \quad A > 0
\]

(1.17)

and

\[
\lambda = (d\beta/dg)|_{g=g_1} > 0.
\]

(1.18)

\( \beta(g) \) has a zero at \( g_1 = (4-D)^{1/2} \cdot \text{const.} = \sqrt{\epsilon} \cdot c. \) \( D \) is the dimension of the transverse space, namely, the dimension of the Reggeon's momentum vector, \( k \). We denote \( 4-D = \epsilon \), and \( \epsilon = 2 \) gives the physical dimension.

Using eqs. (1.16) and (1.17) one immediately sees that the effective coupling constant, \( g(-\ln \xi) \), in eq. (1.15) goes to a fixed value \( g_1 \) as \( \xi \to 0 \); it is then said that the theory has an infrared \( (\xi \to 0) \) stable \( (\lambda > 0) \) fixed point \( (g = g_1) \). If the dimension \( D \) in the theory would have been near \( D = 4 \), then \( g_1 \) would have been small and the perturbation expansion could have been used for \( \Gamma^{m}_{\text{R}}(g) \) in eq. (1.15). Also, the use made of the perturbation theory with only a small number of terms as a guide for studying \( \beta, \gamma, \zeta \) would have been justified a posteriori near \( D = 4 \), because of the smallness of \( g_1 \). However, \( D \neq 4 \); \( k \) is two dimensional. One possibility to continue from here is to follow K. Wilson (see Kogut and Wilson [126]) and expand the results around \( D = 4 \), namely, to expand in power series

![Fig. 16. The \( \beta \) function in Reggeon field theory with a \( \phi' \) coupling \( -g \). An infrared stable fixed point exists at \( g = g_1 \sim \sqrt{4-D} = \sqrt{\epsilon} \).](image-url)
in \( \epsilon = 4 - D \). At the end of the calculation we set \( \epsilon = 2 \) and hope that the series in \( \epsilon \), presumably an asymptotic series, will teach us, if not the reliable values of \( \gamma, \zeta \) etc., at least a lesson about the general structure of the Pomeron's singularity. Note that if, on the other hand, we prefer to work at \( D = 2 \), we have no a priori knowledge of whether the exact \( \beta(g) \) of the theory has an infrared stable fixed point at a small value \( g_1 \), and we have to assume that it is indeed so. (Other difficulties with this theory which exist at \( D = 2 \) are presented in refs. [157 and 173].)

The final expression for \( \Gamma^{n,m}_R \) is given by the following scaling form:

\[
\Gamma^{n,m}_R(E_n, k, \alpha', g, E_N) = E_N \left( \frac{E_n}{\alpha'} \right)^{(2-n-m)D/4} (-E/E_N)^{1-(n+m)\gamma(g_1)/2-z(g_1)(2-n-m)D/4} \times \phi_{n,m} \left( \frac{E_n}{E}, (-E/E_N)^{-z} \frac{k}{E_N} \alpha', g_1 \right)
\]  

(1.19)

where the constants (the "critical exponents") \( \gamma(g_1) \) and \( z(g_1) = 1 - \zeta(g_1) \), are given by power series in \( \epsilon = 4 - D \). In eq. (1.19) the scaling dimensions of \( \Gamma^{n,m}_R \) are different from their physical dimensions. For the non-interacting theory \( \gamma = 0 \), and \( \zeta = 0 \), the physical and the scaling dimensions are, in that case, identical. The anomalous dimensions (\( \gamma, \zeta \neq 0 \)) of \( \Gamma^{n,m}_R \) are entirely due to the Pomeron interactions.

The renormalized Pomeron propagator \( G_{r}^{-1}(E, k) \) will be of major interest in the following. We have here for \( \Gamma^{-1,1}_R = (G_{r}^{-1})^{-1} \) (fig. 15b):

\[
\Gamma^{-1,1}_R(E, k^2) = E_N(-E/E_N)^{1-\gamma(g_1)} \phi_{1,1} \left( (-E/E_N)^{-z} \frac{k^2 \alpha'}{E_N}, g_1 \right).
\]  

(1.20)

\( \phi_{1,1} \) is a function of the scaling variable and will be calculated in section 2. Several interesting results can already be seen from the expressions in eqs. (1.19) and (1.20). The location of the renormalized Pomeron singularity is found at the zero of \( \Gamma^{-1,1}_R \). Even if the exact form of \( \phi_{1,1} \) is unknown, the dependence of the location of the singularity on \( k^2 = -t \) is of the form:

\[
\alpha(t) = 1 + (-t\alpha'/E_N)^{1/z(g_1)} f(g_1).
\]  

(1.21)

If \( z(g_1) \) is really larger than one, as the first loop calculations indicate, then it follows that the slope \( \alpha'(t) \to \infty \) at \( t = 0 \).

From eq. (1.19) the contributions of all \( G^{n,m}_r \) to the forward elastic scattering amplitude \( F(j, 0) \) can be determined. Using eq. (1.2) to calculate \( T(s, 0) \) one finds:

\[
\frac{1}{s} T(s, 0) \sim \sigma_T = N^A N^B (\log s)^{-\gamma} - N_{AB} (\log s)^{-1} + F_{AB} (\log s)^{-2+\gamma} + \cdots.
\]  

(1.22)

The three terms in eq. (1.22) are the contribution to \( \sigma_T \) of the renormalized Green's functions \( G_{r}^{1,1}, G_{r}^{1,2}, G_{r}^{2,2} \), respectively (fig. 17). \( G^{1,1}_r \) gives the leading term if \( -\gamma > 0 \), which is the case here. It is still an open question at this point whether a non-leading term in \( G^{1,1}_r \) dominates the leading term of \( G^{1,2} \) etc. This will be discussed in the next section.

Fig. 17. The sum of all possible diagrams which build the elastic amplitude. At asymptotic energies the sum is dominated by the \( G^{1,1}_r \) contribution.
Table 1
The critical exponents in RFT. Summary of results

<table>
<thead>
<tr>
<th>Method</th>
<th>(-\gamma)</th>
<th>(z)</th>
<th>(\lambda)</th>
<th>(\kappa)</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\epsilon) expansion (O(\epsilon))</td>
<td>0.17</td>
<td>1.08</td>
<td>1</td>
<td></td>
<td>Migdal et al. [139]</td>
</tr>
<tr>
<td>(\epsilon) expansion (O(\epsilon^2))</td>
<td>0.32</td>
<td>1.15</td>
<td>0.34*</td>
<td>0.17</td>
<td>Bronzan and Dash [35]</td>
</tr>
<tr>
<td>Calculation in a (D = 2)</td>
<td>0.11</td>
<td>1.05</td>
<td></td>
<td></td>
<td>Dash and Harrington [71]</td>
</tr>
<tr>
<td>massive (\Delta \neq 0) theory</td>
<td>0.16</td>
<td>1.08</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High temperature expansion</td>
<td>1/2–1</td>
<td>3/2–2</td>
<td></td>
<td></td>
<td>Ellis and Savit [84]</td>
</tr>
<tr>
<td>First loop calculations at (D = 2)</td>
<td>0.124</td>
<td>1.06</td>
<td>1</td>
<td></td>
<td>Frazer et al. [95]</td>
</tr>
<tr>
<td>Upper to three loops ((g^2))</td>
<td>0.13</td>
<td>1.07</td>
<td></td>
<td></td>
<td>Harrington [116]</td>
</tr>
<tr>
<td>calculations</td>
<td>0.2–0.25**</td>
<td>1.13**</td>
<td>0.2–0.3**</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Renormalization group calculations on</td>
<td>1.4–2***</td>
<td></td>
<td></td>
<td></td>
<td>Cardy [62]</td>
</tr>
<tr>
<td>a lattice</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Perturbation theory at large order</td>
<td>0.26 ± 0.02</td>
<td>1.13 ± 0.01</td>
<td>0.49 ± 0.01</td>
<td></td>
<td>Cardy [63]</td>
</tr>
<tr>
<td>Quantum spin model</td>
<td>0.238 ± 0.008</td>
<td>1.16 ± 0.01</td>
<td>0.213</td>
<td></td>
<td>Brower et al. [43]</td>
</tr>
<tr>
<td>(high temp. exp.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Estimated by using Padé approximants (Frazer and Moshe [92]).
** Estimated by using Borel–Padé approximants (Bronzan, Shapiro, Sugar [36]).
*** The method needs refinement in order to provide reliable estimates for the other exponents.

The leading contribution to the elastic scattering amplitude, \(T(s, t)\), can be read from eq. (1.20):

\[
T(s, t) = i N_A(t)N_B(t)(\log s)^{-\gamma} f(t(\log s)^{\delta}).
\]  

(1.23)

\(N_A(t)\) and \(N_B(t)\) are, as in eq. (1.22), the coupling coefficients of the renormalized Pomeron to the hadrons. The scaling function, \(f(t(\log s)^{\delta})\), will be calculated in section 2.

An interesting feature of the result in eq. (1.22) is the asymptotic factorization of \(\sigma_T\) being dominated by the contribution of \(G_R^{-1}\) (fig. 17). (See however ref. [6P] in the Note added in proof.)

The critical exponents \(\gamma\) and \(z\) were calculated by several methods; their results are summarized in table 1.

1.3. Further developments and results

1.3.1. More about renormalization and \(\alpha_{dc}\)

The dimensionless coupling constant in Reggeon field theory is \(g = r\alpha^{D/4} E_N^{(D-4)/4}\), and the theory is super renormalizable for \(D < 4\). However, if we require that the renormalized mass gap \(\Delta = 1 - \alpha\) will vanish, the theory cannot be renormalized order by order in perturbation theory, since the mass renormalization counter term cannot be expanded in power series in \(r_0\). Sugar and White clarified this difficulty in an important study [157] by using a technique analogous to the one used by Symanzik [160] in the \(\phi^4\) theory. They showed how the theory can be formulated in order to provide a complete iterative scheme for calculating its infrared behavior using non-perturbative representations. We will return to this subject in section 2 and discuss these representations, their use in the calculation of the scaling functions, and the approach to scaling. An interesting result (also obtained by Migdal et al.
is that the Pomeron's unrenormalized intercept has to be above one ($\alpha_0 = \alpha_{0c} > 1$) in order that the renormalized Pomeron will be at one. This can be seen in Sugar and White's representation for $\alpha_{0c}$:

$$\alpha_{0c} = 1 + a_0 \int_{0}^{\Lambda^{-1}} \frac{dx}{x} (1 - Z^{-1}(x)). \quad (1.24)$$

$\Lambda$ is a cutoff in the theory; it does not affect the infrared behavior, but it gives a meaning to $\alpha_{0c}$. $a_0 = (r_0^2 \alpha_0^2 D^2)^{1/2}$, and $Z$ is the wave function renormalization constant which satisfies $Z \geq 1$ in the small $\epsilon$ limit.

1.3.2. Secondary trajectories

The effect of Reggeon interactions on meson trajectories was studied [6] in a Reggeon field theory of interacting secondary ($\alpha_R$) and Pomeron trajectories. The important contributions come from diagrams in which one meson trajectory is exchanged accompanied by multi-Pomeron exchanges. One is therefore interested in the effect of the renormalization of the linear bare meson trajectory $\alpha_R(t) = \alpha_R(0) + \alpha_R t$ by RRP and PPP interactions. The modification of the bare meson trajectory is governed by a new critical exponent $\gamma_R$. The amplitude is:

$$T^R(s, t) = N_{AA}^R N_{BB}^R s^{\gamma_R(t)} \exp(-i \pi \alpha_R(t)) + \tau_R (\log s)^{-\gamma_R} \quad (1.25)$$

where, at small $t$, $\alpha_R(t) = \alpha_R(0) + \alpha_R t + O(t^2)$; $\tau_R$ is the signature and $-\gamma_R = 1/12$ in the first order in the $\epsilon$ expansion.

The study of fermion trajectories in Reggeon field theory was carried out by Bartels and Savit [25]. The analysis of fermion trajectories is of particular interest because of the apparent confusion in the Reggeization of fermions. The McDowell symmetry (see, for example, Gribov et al. [107]), imposed by the requirement of Mandelstam analyticity in $\pi N$ scattering, is a symmetry between $u$-channel partial wave amplitudes. It requires the existence of two fermion Regge poles of opposite parity which satisfy $\alpha_+(\sqrt{u}) = \alpha_-(\sqrt{-u})$. Experimentally, however, the baryon trajectory is linear in $u$ and there are no degenerate negative and positive parity baryons. This puzzle was around for many years. Due to Reggeon interaction it was found that a pair of bare fermion trajectories linear in $\sqrt{u}$, which satisfy the McDowell symmetry, are renormalized to a trajectory almost linear in $u$. For $u < 0$, one finds that both negative and positive parity poles exist, but for $u > 0$ only one of the parities remains; the other moves to an unphysical sheet and the puzzling necessity of baryon parity doublets is removed [25].

1.4. A presentation of the content of sections 2 through 9

We end here the brief introduction and short summary of the main early results obtained in RFT. The other important early developments, like the attempts to formulate RFT on a lattice, the analysis of production amplitudes etc., will be mentioned later along with the discussion of recent developments in those topics. The reader who is interested in a more detailed presentation of the topics covered in this introduction may refer either to the original papers or to the existing reviews of the subject by Baker and Ter-Martirosyan [24] on the physical basis of Reggeon calculus, and by Abarbanel et al. [8, 9], in which the motivations, formulation and the early developments in the field are presented. Summer school lectures on the subject were given at Les Houches (White [168]) and McGill University (Moshe [143]), and will be published in the School's Proceedings.
Recently, many interesting developments have taken place in Reggeon field theory, especially in the efforts to elucidate its $s$-channel content and the nature of the ordered phase ($\alpha_0 > \alpha_{0c}$). Formulating the theory on the transverse lattice has contributed much to these important issues. These developments, along with the interesting progress made in the analysis of the production cross sections, cut Reggeon field theory, the calculations of the scaling functions, and the approach to scaling, will be discussed below. Other new developments will be presented as well.

The content of this review is as follows: In section 2 representations of Reggeon Green's functions are derived, using a slightly different method from the one discussed in section 1, for solving the renormalization group equation. These representations are then used in the calculation of the asymptotic form of the forward diffractive peak and the approach to the scaling limit. Then follows a discussion on the attempts to determine the value of the transition energy at which the critical theory is relevant to high energy experiments and of the question of the critical ($\alpha_0 = \alpha_{0c}$) versus non-critical Pomeron.

Recent studies of production amplitudes in the case $\alpha_0 = \alpha_{0c}$ are presented in section 3. The analysis of multi particles production cross sections and the inclusive cross section in the triple Pomeron region shows that past difficulties with a Pomeron intercept $\alpha(0) = 1$, like the Finkelstein–Kajantie problem and the decoupling of the Pomeron, are eliminated from the strong coupling scaling solution in RFT. The Abramovskii, Gribov and Kancheli (AGK) rules are implemented in cut Reggeon field theory as a result of $s$-channel unitarity constraints on the theory. The approach to scaling in the central region is briefly presented.

Section 4, where RFT at zero transverse dimensions is discussed, serves mainly as an introduction to the single site problem in the formulation of RFT on the lattice. Other lessons to be learned from the solution at $D = 0$ or $\alpha' = 0$ are also discussed.

The attempts to apply RFT at the classical tree graph approximation level are analyzed in section 5. After a short discussion of the application of RFT to hadron nucleus scattering, the consistency of the results for hadron hadron with $s$-channel unitarity constraints are checked in the triple Pomeron region and in the central region. The structure of the $\{p, q\}$ phase space at the classical limit is presented, contributing to the understanding of the lattice theory in the following sections.

Then, two path integral approaches to the formulation of RFT on the lattice are discussed in section 6. The difficulties in establishing their correct universality class and in performing numerical calculations are due mainly to the discretization of the rapidity variable as well as to the impact parameter. The high temperature expansion used for the calculation of critical exponents is presented and used again in section 7.3.

In section 7 the formulation of RFT on the impact parameter lattice is presented in detail. It results in the quantum spin analog model of the continuum theory and serves as an extremely useful tool in the very recently made analysis of the ordered phase in RFT ($\alpha_0 > \alpha_{0c}$). The nature of the phase transition in RFT is compared to the more conventional phase transition in the $\phi^4$ theory. The asymptotic grey disc behavior of the cross section is discussed and the consistency of the $n$ particles production cross sections with $s$-channel unitarity constraints is checked in the ordered phase. Two calculations of the critical exponents in the quantum spin model are presented.

In section 8 two very recent works on the nature of the ordered phase are briefly discussed.

Section 9 concludes the review, indicating several issues that need further studies.
2. Representations of Reggeon Green’s functions, the approach to scaling, and the transition energy

At low and medium energies a perturbation scheme for the Pomeron in powers of \( r_0 \) with the first few terms in fig. 17 is adequate for the phenomenological description of the data. As one moves to higher energies, more terms in the series in fig. 17 have a non-negligible contribution to the sum and they have to be taken into account. At high enough \( s \), where the scaling solution of eq. (1.19) is applicable, the amplitude is preferably expressed in terms of the sum of all fully renormalized \( G_R^{1,1} \) (eqs. (1.19) and (1.22)).

The energy at which one hopes to see the onset of the scaling results is of major interest. One can imagine that at some high enough \( s \) we will enter into a transition energy region in which the perturbation expansion in \( r_0 \) of fig. 17 is not useful, since \( s \) is too high and many terms should be taken into account. At this transition region, however, the energy may be too small and the scaling results of eq. (1.19) are not yet applicable. In the transition region, which (as will be discussed below) might be of experimental interest, one would like to understand the smooth transition between the perturbative and scaling solution of Reggeon field theory. At the transition energies, scattering amplitudes can be described either by many terms in the perturbation series or by the scaling solution, including the non-scaling terms which ultimately will vanish as we enter the scaling region (Amati and Jengo [18], Frazer and Moshe [92]). We will choose the second alternative and develop representations for the Reggeon Green’s function useful for the calculation of the terms that govern the approach to scaling. These same representations will be used later to calculate some of the scaling functions in eq. (1.19). The method used here is similar to methods developed by Symanzik [160], Zinn—Justin [173], and Sugar and White [157]. It is a slightly different formalism for the treatment of the renormalization group transformation.

2.1. Representations of Reggeon Green’s functions at \( k_i^2 = 0 \)

We will start with the simplest case and obtain representations at \( t_i = -k_i^2 = 0 \) for the Pomeron propagator and the three-point function. The presentation below follows closely the work of Frazer and Moshe [92]. Similar results were also obtained by Abarbanel, Bartels, Bronzan and Sidhu [7] and will be presented later.

As mentioned above, Reggeon field theory based on the Lagrangian in eqs. (1.9, 11) is a super-renormalizable theory at \( D < 4 \). The standard renormalization scheme is defined at arbitrarily chosen points in phase space. The values of the renormalized quantities at these points are specified in a set of renormalization conditions:

\[
\Gamma_R^{1,1}(E, k^2) \bigg|_{E=0, k^2=0} = 0 \tag{2.1a}
\]

\[
i \Gamma_R^{1,1}(E, k^2) \bigg|_{E=-E_N, k^2=0} = -E_N \tag{2.1b}
\]

\[
i \frac{\partial}{\partial k^2} \Gamma_R^{1,1}(E, k^2) \bigg|_{E=-E_N, k^2=0} = -\alpha' \tag{2.1c}
\]

\[
\Gamma_R^{1,2}(E_1, E_2, E_3, k_i) \bigg|_{k_i \cdot k_i = 0, E_1 = -E_N, E_2 = -E_M = 2E_N, E_3 = -E_1 = E_i, E_1 = E_2 = 1/2E_i, E_1 = 1/2E_i} = \frac{r}{(2\pi)^{(D+1)/2}}. \tag{2.1d}
\]
The renormalization conditions (2.1a–d) differ slightly from the one used by Abarbanel and Bronzan [4]. Equation (2.1a) forces the renormalized Pomeron singularity to have an intercept at one so that we are dealing with the zero mass gap—critical theory. Equations (2.1b) and (2.1c) define the wave function renormalization constant $Z \{\psi_E(x, \tau) = Z^{-1/2} \psi(x, \tau)\}$ and the renormalized slope $\alpha'$. Equation (2.1d) serves as a definition of the renormalized triple Pomeron coupling $r$.

Applying similar technique used for deriving the renormalization group equation (eq. (1.13)), one can derive differential equations for the different renormalization constants in the theory.

There are now three dimensionless parameters in the theory

$$g = \frac{r}{\alpha' D^4} E_N^{-\epsilon/4}, \quad l_2 = \frac{E_M}{E_N}, \quad l_3 = \frac{E_L}{E_N}$$

(2.2)
on which a renormalization constant can depend. Therefore, we have (for example, for the wave function renormalization constant $Z$):

$$\gamma = E_N \frac{\partial \ln Z}{\partial E_N} \bigg|_{r_0, \alpha_0, E_M, E_L}$$

(2.3)

$$= E_N \frac{\partial g}{\partial E_N} \bigg|_{B, E_M, E_L} \frac{\partial \ln Z}{\partial g} + E_N \frac{\partial l_2}{\partial E_N} \bigg|_{B, E_M, E_L} \frac{\partial \ln Z}{\partial l_2} + E_N \frac{\partial l_3}{\partial E_N} \bigg|_{B, E_M, E_L} \frac{\partial \ln Z}{\partial l_3}$$

where $B$ denotes the fact that all derivatives are taken at fixed bare quantities $r_0$, $\alpha_0'$ and fixed $D$.

Equation (2.3) can be simplified by observing that

$$E_M \frac{\partial \ln Z}{\partial E_M} \bigg|_{B, E_N, E_L} = 0 = E_M \frac{\partial g}{\partial E_M} \bigg|_{B, E_N, E_L} \frac{\partial \ln Z}{\partial g} + l_2 \frac{\partial \ln Z}{\partial l_2}$$

(2.4)

and similarly for $E_L$. One then obtains

$$\gamma = \beta \left( \partial \ln Z / \partial g \right)$$

(2.5)

where

$$\beta = E_N \frac{\partial g}{\partial E_N} \bigg|_{B, E_M, E_L} + E_M \frac{\partial g}{\partial E_M} \bigg|_{B, E_N, E_L} + E_L \frac{\partial g}{\partial E_L} \bigg|_{B, E_N, E_M}.$$  

(2.6)

This is just the total derivative with respect to $E_N$ with $l_2$ and $l_3$ held fixed.

$$\beta = E_N \frac{\partial g}{\partial E_N} \left. \left( r_0, \alpha_0', D, E_N, E_M = l_2 E_N, E_L = l_3 E_N \right) \right|_{B, l_2, l_3}$$

(2.7)

$\beta$ is found in perturbation theory. The first loop calculation (Frazer and Moshe [92], Abarbanel et al. [7]) gives the same result as in eq. (1.17), but now $A$ depends on $l_2$ and $l_3$.

$$A = \frac{1}{2} \frac{\Gamma(3-D/2)}{(8\pi)^{D/2}} J(D, l_2, l_3).$$  

(2.8)

As $\epsilon \to 0$ one finds that $J(D, l_2, l_3) \to 3 + O(\epsilon)$. Since in the first loop calculations $\beta(g)$ is determined only up to order $g^3$, and since $g \sim \sqrt{\epsilon}$ near the fixed point, we therefore take only $J = 3$ to this order of the calculations.

The same method used above to derive eq. (2.5) for the wave function renormalization constant can be repeated with any other renormalization constant in the theory. Denote $x_0$ and $x$, an un-
renormalized and renormalized parameter in the theory, then define $Z_x$ from

$$x = Z_x x_0$$  \hspace{1cm} (2.9)

(e.g., $\alpha' = Z_\alpha \alpha_0$, $g = Z_g g_0$, etc.), one finds:

$$\gamma_x = \beta(g) (\partial \ln Z_x / \partial g)$$  \hspace{1cm} (2.10)

where

$$\gamma_x = \frac{d \ln Z_x}{d E_N} (B, E_N, E_i = l_i E_N) |_{B_i}$$

and $\beta$ is given by eq. (2.7). Using the expression for $\beta$ in eq. (1.17) and $\gamma_x(g) = \bar{\gamma}_x g^2 / g^2$, where $\bar{\gamma}_x$ is a constant in the first loop calculation and by combining the solution to eq. (2.10) for $x = g$ and for a general $x$ one finds in first order in $\epsilon$:

$$Z_x = \exp \left\{ \int_0^\epsilon \frac{\gamma_x(g^2)}{\beta(g')} dg' \right\} = \left( 1 + \frac{g_0^2}{g_1^2} \right)^{-2 \bar{\gamma}_x / \epsilon}.$$  \hspace{1cm} (2.11)

Note that $Z_x(g = 0) = 1$, and $g_0 = \{ r_0 / (\alpha')^{D / 4} \} E_N^{-e / 4}$ is the unrenormalized dimensionless coupling constant.

As explained in the introduction the renormalization group equation simply indicates that varying the location of the renormalization point should not affect the physical meaning of the calculations. In other words, the parameters in the theory are correlated in such a way so that the variation $-E_N \rightarrow \eta E_N$ introduces an appropriate change in the parameters and the physical content of the theory remains unchanged. The correlated change in the parameters due to the variation of $-E_N \rightarrow \eta E_N$ can be neatly presented by using eq. (2.11)

$$Z(\eta) = Z(-\eta) \bar{\gamma}_x (g^2 / g^2(\eta))^{-2 \bar{\gamma}_x / \epsilon}$$  \hspace{1cm} (2.12a)

$$\alpha'(\eta) = \alpha'(-\eta) \bar{\gamma}_x (g^2 / g^2(\eta))^{-2 \bar{\gamma}_x / \epsilon}$$  \hspace{1cm} (2.12b)

$$r(\eta) = r(-\eta) \bar{\gamma}_x (g^2 / g^2(\eta))^{-2 \bar{\gamma}_x / \epsilon}$$  \hspace{1cm} (2.12c)

where $Z = Z(\eta = -1)$, $\alpha' = \alpha'(\eta = -1)$, $r = r(\eta = -1)$, and finally,

$$g(\eta) = g_1 \left[ 1 + (-\eta)^{\epsilon / 2} \left( \frac{g_1^2}{g^2} - 1 \right) \right]^{-1 / 2} \rightarrow g_1$$  \hspace{1cm} (2.13)

reflects the existence of the infrared stable fixed point at $g = g_1$.

The advantage of the above method for deriving the scaling relations in eqs. (2.12a–c) and (2.13) is, as will be shown next, that it can be easily extended into a very appealing method for the calculations of the scaling functions and the approach to scaling. Let us first check whether using this method is indeed equivalent to the direct use of the renormalization group equation (eq. (1.13)) and whether it gives the same scaling behavior for $\Gamma_R^{n,m}$ in eq. (1.19). We can calculate $\Gamma_R^{n,m}$ knowing its physical dimensions and using the relations obtained in eqs. (2.12a–c) and (2.13). From dimensional considerations one finds the following general expression for $\Gamma_R^{n,m}$ at the point $E_i = -l_i E_N$, $k_i = 0$:

$$\Gamma_R^{n,m}(E_i = -l_i E_N, k_i = 0, \alpha', r, -E_N) = r^{n+m-2}(-E_N)^{3-m-n} \phi_m(g, l_i)$$  \hspace{1cm} (2.14)

where the last parameter $-E_N$ of $\Gamma_R^{n,m}$ denotes the fact that the theory was renormalized employing the renormalization conditions at the phase space points shown in eqs. (2.1a–d). The relation between
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F and the unrenormalized single particle irreducible proper vertex $F_{mn}$ gives:

$$iF_{mn}(E_1 = lE, k_1^2 = 0, \alpha', r, -E_N) = Z(g)^{m+n/2} iF_{mn}(E_i = lE, k_i^2 = 0, \alpha_i, r_0)$$

$$= Z(g)^{m+n/2} Z(g(\eta))^{-(m+n)/2} i\Gamma_R^{m,n}(E_i = \eta E_N, k_i^2 = 0, \alpha', r, \eta E_N)|_{\eta = E/E_N}. \quad (2.15)$$

Using eq. (2.14) one finally finds:

$$i\Gamma_R^{m,n}(E_i = lE, k_i^2 = 0, \alpha', r, -E_N) = \frac{2}{E} \sqrt{2} \{\frac{g^2}{g_1^2} \left[ 1 + \left( -\frac{E}{E_N} \right)^{\gamma_g} \left( \frac{g_1^2}{g^2} - 1 \right) \right] \}^{(m+n)/\nu} \gamma_{m+n}/\nu - \left( 2\nu + 1 \right) (m+n-2)$$

$$\times \left\{ g_{m+n}/\nu - \left( 2\nu + 1 \right) (m+n-2) \right\} \gamma_{m+n}/\nu - \left( 2\nu + 1 \right) (m+n-2) (-g)^{m+n-2} \phi_{m,n}(g, l). \quad (2.16)$$

Equation (2.15) is, of course, similar to eq. (1.19), which was the solution to the renormalization group equation. In eq. (2.15), however, non-leading terms were explicitly presented; they are present if the physical renormalized coupling constant $g$ does not equal its fixed point value $g_f$. Thus, the terms governing the approach to scaling can be analyzed. The other interesting benefits derived from the present formulation will become more apparent later when the form of the scaling functions will be derived.

At $k_1^2 = 0$ the functions $\phi_{m,n}(g, l)$ are unknown except for $\phi_{1,1} = 1$ and $\phi_{1,2} = (2\pi)^{-(D+1)/2}$, which are determined from the renormalization conditions in eqs. (2.1b) and (2.1d). In the first loop calculations one finds from eq. (2.11):

$$g = g_0 \left( 1 + \frac{g_0^2}{g_1^2} \right)^{-2\gamma_g/\nu} \quad (2.17)$$

where $\gamma_g = \beta(g)/g + \epsilon/4$ is evaluated at $g = g_1$ ($\gamma_g = \epsilon/4$), and finally, using eq. (2.17) together with (2.16), the representations for the Pomeron inverse propagator $\Gamma^{1,1} = (G^{1,1})^{-1}$ and the three-point proper vertex $\Gamma^{1,2}$ are obtained (we denote from here on $\gamma = \gamma_g$ and $z = 1 - \zeta$):

$$i\Gamma^{1,1}(E, 0) = E \left[ 1 + \kappa(-E)^{-\epsilon\nu/\gamma} \right] \gamma_{1,1}/\nu - \left( 2\nu + 1 \right) (1-\zeta) \nu\gamma$$

$$i\Gamma(E, l_2 E, l_2 E, k_2^2 = 0) = \left[ 1 + \kappa(-E)^{-\epsilon\nu/\gamma} \right] (3\nu)(1-\zeta) \nu\gamma$$

$$\quad \times \left\{ \frac{g_0^2}{g_1^2} \left[ 1 + \left( -\frac{E}{E_N} \right)^{\gamma_g} \left( \frac{g_1^2}{g_0^2} - 1 \right) \right] \}^{(m+n)/\nu} \gamma_{m+n}/\nu - \left( 2\nu + 1 \right) (m+n-2) \right\} \gamma_{m+n}/\nu - \left( 2\nu + 1 \right) (m+n-2) (-g)^{m+n-2} \phi_{m,n}(g, l_i). \quad (2.18)$$

where

$$\kappa = r_0^2 [\alpha_{DP}^2 g_1^2]. \quad (2.20)$$

The Mellin transform (eq. (1.2)) of eq. (2.18) will give us the factorizable contribution of $\Gamma^{1,1}$ to $\sigma_T$.

$$\sigma_T = N_A N_B \Phi(-\frac{1}{6}, 1; -\kappa \log s). \quad (2.21)$$

$\Phi(a, b; z)$ is the confluent hypergeometric function (sometimes called $\phi(a, b; z)$). This compact representation of $\sigma_T$, calculated to the first loop order (Frazer and Moshe [92]), can be represented either in its power expansion or in its asymptotic expansion (see A. Erdélyi—Bateman manuscript project [86]), namely,

$$\Phi(-\frac{1}{6}, 1; -\kappa \log s) = \left\{ \begin{array}{ll}
\text{"Perturbation expansion"} & \sim \sum_{n=0}^{\infty} a_n (r_0^2 \log s)^n \\
\text{Asymptotic expansion} & \sim (\log s)^{1/6} \sum_{n=0}^{\infty} b_n (\kappa \log s)^{-n}.
\end{array} \right. \quad (2.22a)$$

Equation (2.22) represents the "dual languages" (Amati and Jengo [18]) for the Pomeron singularity above and below the transition energy discussed above. Namely, a "perturbation expansion" (2.22a) in
powers of \( r_0^2 \) at low \( s \) which turns into the scaling result plus corrections to scaling (2.22b) at higher \( s \). Note, however, that eq. (2.22) is obtained in the \( \epsilon \) expansion and that the "perturbation expansion" is actually an expansion in \( r_0^2 \ln s \) of the scaling solution. It correctly reproduces the order \( r_0^2 \) of the perturbation expansion; this reflects the fact that eq. (2.21) was obtained in the first loop approximation. Higher order terms in the expansion (2.22a) give the leading \( \epsilon \) terms in each order of the perturbation [157, 92, 7].

\( \sigma_T \) can be rewritten in terms of renormalized quantities:

\[
\sigma_T(s) = N_A N_B \left( \frac{g_1}{g} \right)^{2/16} (E_N \ln s)^{d/12} \left\{ \Gamma^{-1} \left[ 1 + \frac{\epsilon}{12} \left( \frac{g_1 - g}{g_1} \right)^2 \Gamma^{-1} \left( 1 - \frac{5}{12} \epsilon \right)(E_N \ln s)^{-d/2} \right] - \frac{5}{72} \left( \frac{g_1 - g}{g^2} \right)^2 \Gamma^{-1} \left( 1 - \frac{11}{12} \epsilon \right)(E_N \ln s)^{-d/2} \right\}. \tag{2.23}
\]

2.2. The general form of the non-leading terms

The leading and non-leading contributions of \( G_{R}^{1,1} \) in eq. (2.23) were calculated only in first order in \( \epsilon = 4 - D \). From a phenomenological viewpoint (as will be discussed below) as well as from a theoretical one it is interesting to know the general form of the terms that govern the approach to scaling (Moshe [141]). We can follow the method described above without limiting ourselves to a certain order in the perturbation expansion. We assume, however, that an infrared stable fixed point exists at \( g = g_1 \). For any renormalization constant we now have:

\[
Z_x = \exp \left\{ \int g \frac{\gamma_x(g')}{\beta(g')} \, dg' \right\} = f_x(g) \left( 1 - \frac{g}{g_1} \right)^{\gamma_x(g_1)/\lambda} \tag{2.24}
\]

where \( f_x(g) \) is a smooth function of \( g \) that satisfies \( f_x(0) = 1 \) and where

\[
\lambda = d \beta/dg \bigg|_{g = g_1} > 0 \tag{2.25}
\]

will play the role of a new critical exponent (called sometimes "sub-critical" exponent – Brezin, Le Guillou and Zinn Justin [32]). In particular, for \( Z_x = Z_g \) we have:

\[
g = g_0 Z_g = g_0 f(g)(1 - g/g_1)^{\gamma_x(g_1)/\lambda} \tag{2.26}
\]

and therefore for any \( Z_x \)

\[
Z_x(g) = f_x(g)f(g)^{-\gamma_x(g_1)/\lambda} = \phi_x(g)(g/g_0)^{\gamma_\lambda/\beta} \tag{2.27}
\]

where

\[
\gamma_\lambda(g_1) = -P \left[ \frac{1}{g} \frac{\beta(g)}{g = g_1} \right] = -P = \frac{\epsilon}{4}. \tag{2.28}
\]

\( P \) is the power of \( E_N \) in the dimensionless coupling constant \( g_0 = r_0 a_0^{-D/4} E_N^P \) (\( P = -\epsilon/4 \)). The infrared behavior of \( \Gamma^{m,m} \) is determined as before by \( Z_x(g(g_0)) = Z_x(E_N) \) as \( E_N \to 0 \). The functions \( f_x(g) \) and \( f(g) \) determine the "non-universal" coefficient which governs the approach to scaling. Their explicit form, which can be calculated, for example, in an \( \epsilon \) expansion, does not change any of the "universal" parameters. Given \( f_x(g) \) and \( f(g) \) we would like to find the functional dependence of \( \Gamma^{m,m} \) on these functions. Equation (2.26) can be solved iteratively for \( g(g_0) \) which will then be used in eq. (2.27) to determine \( Z(E_N) \) – the wave function renormalization constant. A second order iteration gives for
iG_{R}^{1,1}(E,0) = Z^{-1}(E_{N}) iG_{R}^{1,1}(E,0) = (Z(-E)/Z(E_{N})) (-E)^{-1}:
\[iG_{R}^{1,1}(E,0) = \frac{\phi(g_1)}{\phi(g)} \left( \frac{g_1}{g} \right)^{y-P} (-E)^{-1} \left( \frac{-E}{-E_{N}} \right)^{\gamma} \times \left[ 1 + ha_1(g_1 - g) \left( \frac{-E}{-E_{N}} \right)^{\lambda} + h^2 a_2(g_1 - g)^2 \left( \frac{-E}{-E_{N}} \right)^{2\lambda} + O \left( \left( \frac{-E}{-E_{N}} \right)^{3\lambda} \right) \right]. \quad (2.29)\]

\(\phi(g) = \phi_2(g)\) is defined in eq. (2.27) and \(h\) and \(a_n\) are given functions of \(f_2(g)\) (Moshe [141]). The functions \(a_n = a_n(f, f_2)\) are distinguished by being exactly defined by an iteration of order \(m \leq n\) in the solution of eq. (2.26) and therefore the second order iteration is enough to determine \(a_1\) and \(a_2\).

The representation of \(\sigma_T\) in eq. (2.23) is a special case of eq. (2.29), where \(\phi\), \(h\), \(a_1\), \(a_2\), and the critical exponents were calculated in first order in \(\epsilon\),
\[a_1 = \frac{\gamma}{P} = \frac{1}{3}, \quad f(g) = (1 + g/g_1)^{1/2}\]
\[a_2 = 4 \epsilon \frac{2}{\gamma} \left( \frac{2}{\gamma} + 1 \right) = -\frac{5}{18}, \quad f_2(g) = f(g)^{2y_\lambda} = f(g)^{-1/3}, \quad \phi = 1 \quad (2.30)\]
\[(g_1/g)^{-y_P} = (g_1/g)^{-1/3}, \quad h = \frac{1}{2g_1} (g_1/g)^2 (1 + g/g_1).\]

The contribution of \(G^{1,1}\) to \(\sigma_T\) has the general form:
\[\sigma_T = A_0 (\log s)^{-\gamma} [1 + (g - g_1) A_1 (\log s)^{-\lambda} + (g - g_1)^2 A_2 (\log s)^{-2\lambda} + O((\log s)^{-3\lambda})]. \quad (2.31)\]

Reggeon field theory is a super-renormalizable theory for \(D < 4\) space dimensions and therefore the approach to scaling here is with power of \((-E)^{\lambda}\), as shown in eq. (2.29) rather than with \((\log E)^{-\lambda}\). This is different from the logarithmic approach to scaling (e.g., in the ultra-violet \(q^2 \to \infty\) region) in nonabelian gauge theories in \(d = 4\) space time dimensions and is similar to \(\phi^3\) in \(d = 4\) space time dimensions where the approach to scaling is with powers of \(q^2\). The new critical exponent \(\lambda\) has been calculated in the \(\epsilon\) expansion, the results are [23, 35, 92]:

one loop \(\lambda^{(1)} = \frac{1}{2}\epsilon\)
\[\lambda^{(2)} = \frac{1}{2}\epsilon - \epsilon^2 \left( \frac{157}{576} + \frac{149}{288} \log^4 \frac{4}{3} \right). \quad (2.32)\]

\(\lambda\) has a poorly convergent expression. The critical exponent \(\lambda\) is well known in solid state physics (see Wagner [163]; Brezin, Le Guillou and Zinn-Justin [32]), where interestingly it has also one of the worst-converging \(\epsilon\) expansions; it was calculated to order \(\epsilon^3\) and evaluated by using Padé's approximants. From eq. (2.32) we find that \(\lambda(\epsilon = 2) < 0\), whereas Padé's approximants suggest \(\lambda(\epsilon = 2) = 0.37\) and higher orders in \(\epsilon\) might change this value considerably. \(\lambda\) can be estimated also from the three loops calculations (\(\lambda = 0.2-0.3\), see table 1 and section 4).

The non-leading terms in eq. (2.31) have to be compared with the leading contributions from \(G^{1,2}\) and \(G^{2,2}\) etc. in order to determine the most important corrections to the \((\log s)^{-\gamma}\) behavior of \(\sigma_T\). Looking back to eq. (1.22) we see that \((\log s)^{-\gamma - \lambda}\) of eq. (2.31) should be compared with \((\log s)^{-1}\) and \((\log s)^{-2+\gamma}\). If \(-\gamma\) is indeed of order 0.3 and \(\lambda \sim 0.3-1.0\), then the non-leading term \((\log s)^{-\gamma - \lambda}\), and maybe also \((\log s)^{-2+\gamma}\) from eq. (2.31) are more important than the contribution of \(G^{1,2}\) and \(G^{2,2}\) in eq. (1.22).
It was shown above that in the $\epsilon$ expansion Reggeon field theory possesses an infrared stable fixed point. Therefore the infrared limit ($E \to 0$) or equivalently, the high energy ($s \to \infty$) behavior of the theory, depend only on the value of its parameters at the fixed point. Other Pomeron interactions were shown therefore to be unimportant as $s \to \infty$; but although they do not change the infrared asymptotic behavior of the theory, they certainly are important in considering the approach to this limit. The four Pomeron interaction $\lambda_1 \bar{\psi}^2 \psi^2 + \lambda_2 (\bar{\psi} \psi^3 + \bar{\psi}^3 \psi)$, which is infrared free (Abarbanel and Bronzan [5]), gives an asymptotic constant contribution to $\sigma_T$, which has to be compared with $(\log s)^{-\gamma - \lambda}$. As long as the values of $-\gamma$ and $\lambda$ are not known well enough, it is difficult to tell who dominates the non-leading contributions.

A reliable calculation of the exponents with an accuracy of 10–15% will be of much phenomenological interest.* As will be discussed below it seems that the ISR energy region might be reasonably described by the scaling solution accompanied however by the terms that govern the approach to scaling. Methods used for RFT on a lattice may be powerful enough to give the critical exponents at such an accuracy (see section 7.3). Having accurate theoretical values for the exponents $\gamma$, $z$ and the new exponent $\lambda$, it is possible to find the non universal constants that determine the approach to scaling by fitting the data (e.g., $A_1$, $A_2$ in eq. (2.31) and others in elastic and inelastic scattering). A discussion of the energies at which such a practical approach to RFT might be relevant is given in section 2.4.

We will return now to a further exploration of the method described in section 2.1 to find representations for Reggeon Green's functions and use them for calculating the slope of the diffraction peak and the approach to its asymptotic scaling form.

2.3. Representations of Reggeon Green's function at $k_i^2 \neq 0$

The derivation of the representation at $k_i^2 = 0$ has been described in detail in section 2.1 and it is easy to see now how to proceed and obtain representations for $k_i^2 \neq 0$, from which the slope of the diffraction peak can be determined. The method for calculating the scaling functions will be presented here following again the work of Frazer and Moshe [92]. Similar results, obtained by Abarbanel et al. [7], using a slightly modified method, will be discussed later.

In order to derive representations for $k_i^2 \neq 0$, the renormalization condition in eq. (2.1b) is replaced by:

$$i \Gamma_R^{1,1}|_{k = -E_N} = -E_N - \alpha' k_N^2. \quad (2.33)$$

$\alpha'$ is defined by the derivative with respect to $k^2$ of $\Gamma_R^{1,1}$ at $E = -E_N, k = k_N$ (instead of eq. (2.1c)), $\Gamma_R^{1,2}$ is defined at $k_i = k_N$, $k_2 = v_2|k_N|$ and $k_3 = v_3|k_N|$. Note that in addition to the dimensionless parameter $g = r\alpha'^{-\frac{3}{4}}E_N^{-\frac{1}{4}}$, we have also $h = \alpha' k_N^2/E_N$.

In general, if there are $g$, $h$, $y_3$, $y_4$, ... dimensionless parameters, the equivalent equation to eq. (2.10) will be:

$$\left| \begin{array}{c} \gamma_x \\ \xi_x \\ \xi_{h} \end{array} \right| = \left| \begin{array}{c} \beta_E \\ \beta_k \\ \beta_h \end{array} \right| \left| \begin{array}{c} \frac{\partial}{\partial g} \ln Z_x \\ \frac{\partial}{\partial k} \ln Z_x \\ \frac{\partial}{\partial h} \ln Z_x \end{array} \right| \left| \begin{array}{c} \xi_{y_1} \\ \xi_{y_2} \\ \xi_{y_3} \\ \xi_{y_4} \end{array} \right| \sum_{y_1, y_2, y_3, y_4} \left( \begin{array}{c} \xi_{y_1} \\ \xi_{y_2} \\ \xi_{y_3} \\ \xi_{y_4} \end{array} \right) \frac{\partial \ln Z_x}{\partial y} \quad (2.34)$$

where

$$\xi_x = E N \left. \frac{d x}{d E_N} \right|_N \quad \text{and} \quad \xi_{k} = k_N^2 \left. \frac{d x}{d k_N^2} \right|_N$$

and similar equations can be derived for $(\partial \ln Z_x/\partial h)$ and $(\partial \ln Z_x/\partial y)$.

*See the Note added in proof on recent calculations in refs. [63] and [43] and the last entries in table 1 (section 1).
In our case we have $g$ and $h$ only; furthermore, if we restrict ourselves to solving the problem to first order in $\epsilon$, then the equation for $\partial \ln Z/\partial h$ has contributions from order $\epsilon^2$ only. Equation (2.34) can be then solved to obtain the representation for the wave function renormalization constants $Z$ and for $Z_\alpha$, from which one finds $G^{1,1}(E, k)$ as before (eqs. (2.18) or (2.29)),

$$-[i\Gamma^{1,1}_{(E, k)}]^{-1} = iG^{1,1}(E, k) = \frac{-Z(-E, k)}{E - \alpha'(-E, k)k^2}$$

$$iG^{1,1}(E, k^2) = \{-E + \alpha'_0 k^2[1 + \kappa(-E)^{-\epsilon^2}]-[1 + \kappa(-E)^{-\epsilon^2}]^{-2}\epsilon\}. \tag{2.36}$$

Equation (2.36) reduces, of course, to eq. (2.18) for $k^2 = 0$.

The elastic scattering amplitude is obtained from the Mellin transform of eq. (2.36)

$$\frac{1}{s} T(s, t) = N_A(t)N_B(t) \sum_{n=0}^{\infty} \left( \alpha'_0 t \log s \right)^n n! \Phi \left( \frac{1}{6} - \frac{n}{12}, n + 1; -\kappa \log s \right) \tag{2.37}$$

where $\Phi(a, b; z) = _1F_1(a, b; z)$ and at small values of the scaling variable

$$x = \frac{\alpha'_0 t}{E_N} (E_N \log s)^{1+\epsilon/24}. \tag{2.38}$$

A consistent $\epsilon$ expansion of eq. (2.37) (in terms of the renormalized parameters) gives:

$$\frac{1}{s} T(s, t) = N_A(t), N_B(t)(E_N \log s)^{\epsilon/12} \exp \left\{ \frac{\alpha'_0 t}{E_N} (E_N \log s)^{1+\epsilon/24} \right\} \times \left\{ 1 - \frac{\epsilon}{12} \phi_1(x) \right.$$  

$$+ \frac{1}{12} \left( \frac{g_2^2 - g_1^2}{g_2^2} \right) (E_N \log s)^{\epsilon/2} \left( \phi_2(x) - \frac{\epsilon}{24} \phi_3(x) \right) + O((g_2^2 - g_1^2)^2 \log s^{-\epsilon}) \right\}. \tag{2.39}$$

At $t = 0$ eq. (2.39) reduces to the $\epsilon$ expanded version of eq. (2.23). The functions $\phi_i(x)$ are smooth functions of the scaling variable and are evaluated from eq. (2.37) (Frazer and Moshe [92]). The functions $N_A(t)$ and $N_B(t)$ are exponentials in $t$. Equation (2.37) presents the contribution of the renormalized Pomeron propagator to the diffraction peak. The series is an asymptotic series approaching an exponential in $x$ at small values of this scaling variable. At large values of $x$ there is a dip and a second maximum which is several orders of magnitude smaller than the amplitude at $t = 0$. However, the location of the dip and the height of the second maximum vary as powers of $\log s$. This is due to the fact that we reach values too high for the scaling variable beyond the range of validity of eq. (2.37). At small values of $x$, the $\epsilon$ expanded form in eq. (2.39) is also approximately an exponential with no dip structure [92].

Abarbanel, Bartels, Bronzan and Sidhu [7] used a modified version of the method described above. The renormalization condition [4]

$$i (\partial \Gamma_{R}^{1,1}(E, k^2)/\partial E) \bigg|_{k^2 = 0} = 1 \tag{2.40}$$

was used, instead of eq. (2.1b), and therefore the Green's functions are given by an integral over the wave function renormalization constant $Z(\eta)$ (see also Sugar and White [157]), rather than by a simple product as in eq. (2.15). A consistent matching procedure was employed for the residues of the poles at $g = g_1$. The $\epsilon$ expansion was employed for the renormalization group functions $\gamma, \beta, \eta$ in eq. (2.34) rather than for the whole equation and therefore $\partial \ln Z/\partial h \neq 0$ in first order in $\epsilon$. This last step extends the region at which the method is applicable to values of the scaling variable $x$ higher than in [92].
The result obtained in ref. [7] for the diffractive peak given by the leading scaling function $F_1(x)$ reveals an interesting structure seen in fig. 18. The height of the secondary maximum relative to the $t = 0$ value is independent of $\alpha'_0$ and $\rho_0$ and is $\sim 10^{-6}$. To fix the location of the dip, however, one has to set the scale for $s$ and $t$ which results in fixing $\alpha'_0 = 0.5$. $F_1^2(x)$ gives the gross features of the diffraction peak at the ISR energies (White [168]) with almost no need however for the $t$ dependence in $N_A(t)N_B(t)$. Although this fit cannot be taken too seriously at present energies, obtaining the prediction of the ratio $10^{-6}$ between the forward peak and the secondary maximum is quite impressive. This result is of interest in particular in view of the discussion on the universal behavior of the fixed point theory in section 1; namely, once we assume that $\alpha(0) = 1$ and the existence of the infrared stable fixed point, the gross features of results such as fig. 18 are independent of the physical values of the parameters. The parameters $\rho_0$ and $\alpha'_0$ appear only in constants that fix the scales; the theory has an asymptotic large energy behavior depending only on the fixed point values of the renormalized parameters.

Note that a diffractive peak of the form in eq. (2.39) will finally give at very high $s$ (Migdal et al. [139])

$$\sigma_{el} \sim \sigma_{el}^2 \langle \log s \rangle^{-(1-\zeta)D/2} \sim \langle \log s \rangle^{-2\gamma - (1-\zeta)D/2}$$

and the first order $\epsilon$ expansion gives $-2\gamma - (1-\zeta)D/2 = -5/6$, namely, $\sigma_{el}/\sigma_{tot} \to 0$ as $s \to \infty$. From bounds obtained in lattice formulation of RFT Cardy and Sugar [57] had shown that indeed at $D=2 - \gamma - (1-\zeta) = 0$ and therefore $\sigma_{el}/\sigma_t \to 0$ (see section 6).

The representations of the Reggeon Green's functions obtained by Abarbanel et al. [7] and Frazer and Moshe [92] were evaluated in the $\epsilon$ expansion. The method was further developed by Frazer et al. [95] and employed to obtain representations directly at the physical number of dimensions ($D = 2$). The ultraviolet cutoff in $k^2$, which was attached to the "bare" Pomeron propagator

$$iG^{1,1}(E, k) = \frac{e^{-bk^2}}{E - \alpha'_0 k^2 - \Delta_0}$$

and

$$F_1^2(x) = \left( \frac{\log s}{s} \right)^{x-x_0}.$$
is an “irrelevant” variable and does not affect the infrared behavior of the theory. It prevents, however, divergences in the perturbation theory at $D = 2$ and gives meaning to the intercept shift (eq. (1.24), see also Wallace [164]), which is evaluated nonperturbatively. The cutoff slightly affects the approach to the scaling limit; e.g., the total cross section behaves like

$$
\sigma_T \sim (\ln s)^{-\gamma \left(1 + (\ln s)^{-\lambda}(C_0 + C_1 \ln(\ln s)) + O((\ln s)^{-2\lambda})\right)}
$$

(2.43)

where $C_1$ is proportional to $b$.

The $t$ dependence of the diffractive peak was calculated also in the first loop order directly at $D = 2$ [95], thus avoiding the approximation involved in the $\epsilon$ expansion. The result is not much different than that of the $\epsilon$ expansion calculation of Abarbanel et al. [7] in fig. 18, and the first term that governs the approach to scaling ($- (\log s)^{-\lambda f}(x)$) is small compared to the leading term at large $s$. Other forms of cutoffs related to threshold effects were considered in [72] and [75].

2.4. Evaluation of the transition energy

The energy at which the Reggeon field theory scaling results become phenomenologically valuable is of major importance and its evaluation was studied by several groups. We will see below, however, that there are several theoretical and phenomenological difficulties which unfortunately have not yet been solved, preventing us from obtaining a reliable value for this energy. The source of these difficulties will be discussed in this section. The problem can be approached from two different viewpoints: (a) through the analysis of the perturbation series (Migdal et al. [139], Capella and Kaplan [53], Kaidalov and Ter-Martirosyan [124]), and (b) through the analysis of the approach to scaling (Amati and Jengo [18], Frazer and Moshe [92], Abarbanel et al. [7], Frazer et al. [95]). With some variation among the various groups the general picture is as follows: At medium energies one expects that a perturbation expansion in powers of $r_0$ would be adequate for describing the effect of Pomeron interactions. Above a certain transition energy $s_T$, as discussed above; a more adequate description of the effect of Pomeron interaction is given by the scaling solution presumably accompanied by a few terms that govern the approach to scaling. These two equivalent schemes are “dual” to each other in the sense that both of them result from the same physics, namely, Pomeron interactions, and describe the same phenomena at different energies. Amati and Jengo [18] expressed the opinion that it is reasonable to expect this “duality” to be a general feature of the theory. The gross features of such schemes are realized although in a much more simplified manner by the $\Phi$ function in eq. (2.22). A more delicate analysis that will include other nonleading contributions (section 2.2) is certainly needed.

A possible method for estimating the transition energy $s_T$ uses the Pomeron Green’s function representation in eq. (2.18). The Mellin transformation (eq. (1.2)) of its inverse gives:

$$
\sigma_T = N_A N_B \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \left[1 + \frac{r_0^2}{\alpha_0^2 g_1^2} (-E)^{-2} \right]^{-2+\epsilon} e^{-E \gamma} \frac{dE}{E}.
$$

(2.44)

The transition from a power expansion in $r_0^2 \ln s$ to an asymptotic expansion of the form $(\ln s)^{-\gamma (1 + (g - g_1) a(\ln s)^{-1} + \cdots )}$ occurs at an energy determined by the parameter $E = E_0 = r_0^2 g_1^{-2} \alpha_0^{-1}$. In the first loop approximation in which eq. (2.44) was obtained one finds at $D = 2$ that $g_1^2 = 8 \pi / 3$, and therefore $E_0 = 3 r_0^2 / 8 \pi \alpha_0^3$. An estimate of the transition energy is then found by equating

$$
\ln(s_T/s_0) = \frac{1}{E_0} = g_1^2 \alpha_0' (1/r_0^2) = 8 \pi \alpha_0' / 3 r_0^2
$$

(2.45)
where the mass scale is $s_0 \sim 1 \text{GeV}^2$. The transition energy $s_T$ is therefore directly related to the experimentally measured value of the bare triple Pomeron coupling and bare Pomeron slope $\alpha'_0$.

The logic of estimating the transition energy from eq. (2.44) [18, 92, 7, 95], is somewhat different from that of an estimate based on perturbation theory alone [139, 54, 124]. The question asked above was: At what low energies the asymptotic scaling solution stops to be reliable? or more precisely, at what low energies the terms governing the approach to scaling are of considerable size? The answer to this question (eq. (2.45)) depends on $g_\alpha$, which characterizes the scaling solution, as well as on the bare parameters $r_0$ and $\alpha'_0$. The transition energy can be also expressed in terms of renormalized parameter only, as seen in eq. (2.23). The alternative question based on perturbation theory alone is: At what high energies would the typical expansion parameter $(g_\alpha/8\pi\alpha'_0) \ln s$ be large enough so that many terms in the perturbation series would be needed in order to describe the data? There is some confusion in the literature as to the notations of what is meant by the triple Pomeron coupling. Table 3 [96] summarizes and compares the different definitions. Naturally, the expansion parameter, $(g_\alpha/8\pi\alpha'_0) \ln s$, does not depend on any renormalized parameter from the scaling solution and is slightly different from eq. (2.45). It implies that perturbation expansion is of interest at energies at which $\log s < \log s_T = 4\pi\alpha'_0/r_0$.

The expressions given above for $\log s_T$ and $\log s_p$ should be considered however only as typical values and certainly not as rigorous methods for their evaluation. In general, the energies $s_T$ and $s_p$ differ from one another since different contributions dominate the perturbation series and the asymptotic solution. (The possibility that the two regions overlap, namely, that $s_T < s_p$, is very attractive.) In the simple example of eq. (2.22) the value of $s_p$ is not determined properly since the perturbation expansion is reproduced only to order $r_0^2$ and only the $G_{1,1}$ contribution is included. Also, note that in general the approach to scaling involved the exponent $\lambda$, and the value $\lambda = 1$ used in eq. (2.45) is calculated in the first order in $\epsilon$ only. Bearing in mind the limitations of the methods for defining these energies, we will proceed by discussing their numerical estimates.

A reliable value of $r_0$ can be extracted from the recent measurement of the FNAL-Single Arm Spectrometer Group [88]. The inclusive cross section was measured in the triple Pomeron region (Feynman $x$ between 0.975 < $x_F$ < 0.995) at $s = 265$ and 330 GeV$^2$ for $pp \rightarrow px$, $\bar{p}p \rightarrow \bar{p}x$, $\pi^+p \rightarrow \pi^+x$ and $K^+p \rightarrow K^+x$. The triple Pomeron diagram (fig. 7) in $ap \rightarrow ax$, dominating the above phase space region, contributes [96]

$$M^2 \frac{d\sigma}{dt \, dM^2} = \beta_p(0)\beta^2_p(t) \frac{\sqrt{2} r_0}{16\pi} \left( \frac{s}{M^2} \right)^{2\alpha(t) - 2} (M^2)^{\alpha'(0) - 1}. \tag{2.46}$$

The slow variation of the cross section with $M^2$ due to the smallness of $\alpha'_0$ in eq. (2.46) describes the data well (see for example fig. 19). The ratio

$$M^2 \frac{d\sigma}{dt \, dM^2}/d\sigma_{el}/dt = \beta_p(0) \frac{\sqrt{2} r_0(t)}{\beta^2_p(t)} \tag{2.47}$$

in table 2 is clearly independent of the projectile, as it should be, and is consistent with a small $\sigma'_0$. If one identifies $r_0(0) = r_0$, then, from table 2, eq. (2.46) and the elastic scattering data [88], one finds $r_0 = 0.59 \pm 0.05 \text{GeV}^{-1}$. This gives in eq. (2.45) an estimate for the transition energy (using for $\alpha'_0 = 0.3 \pm 0.1$ a range of values found from fits to the FNAL [88] and ISR data [124])

$$Y_T = \log s_T = 7.2 \pm 2.4. \tag{2.48}$$

The above value for $s_T$ should however be considered with appropriate caution for several important
M. Moshe. Recent developments in Reggeon field theory

**Fig. 19.** The inclusive cross section for $pp \rightarrow px$ measured by the Fermilab Single Arm Spectrometer Group [88]. The same $M^2$ independence of $M^2 \frac{d\sigma}{dM^2 dt}$ is also concluded from the measurement of $\pi^- p \rightarrow \pi^- x$, $k^- p \rightarrow k^- x$, and $pp \rightarrow px$. The incident hadron momenta were $P_{\text{lab}} = 150 \text{ GeV/c}$ and $175 \text{ GeV/c}$ in this experiment.

reasons: (a) Note that $g_1^2$ in eq. (2.45) was calculated in the first order $\epsilon$ expansion. The calculation of Frazer et al. [95] at $D = 2$ gives a smaller $g_1^2$ by a factor of $\frac{3}{4}$ and higher orders may vary as well. (b) Ignoring the multi-Pomeron couplings at small $s$ and the $t$ dependence of the various couplings may also result in a considerable effect on the evaluation of $s_T$. (c) Note that in $s_T \sim r_0^{-2}$ and a slight change in the value of $r_0$ will severely affect $s_T$. (d) The value of $r_0$ extracted from eqs. (2.46) or (2.47) depends on $\beta$, which in turn depends on the parametrization of $\sigma_T$ or $\sigma_{el}$. (e) Although we used the value of $r_0$ from the FNAL data [88] it should be noted that the ISR data predicted a somewhat lower value for $r_0$ [53, 124, 132]. (f) Capella, Kaplan and Tran Thanh Van [55] argued against the use of eq. (2.46) for extracting the value of $r_0$ from the data, since the absorptive corrections to this expression are very large. It is not surprising, then, after considering (a)–(f), to find that different estimates of $r_0$ differ in extreme cases by as much as a factor of 3. In fact, Kaidalov and Ter-Martirosyan [124] found (their $r = r_0/\sqrt{4\pi}$, see table 3) that $r_0 = 0.18 \pm 0.04 \text{ GeV}^{-1}$. Noting that this implies a factor of $-10$ for

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$A$ (mbGeV$^{-2}$)</th>
<th>$B$ (GeV$^{-3}$)</th>
<th>$C$ (GeV$^{-4}$)</th>
<th>$M^2 \frac{d^2\sigma}{d\Omega dM^2}$ = $A \exp(Bt+Ct^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^- $</td>
<td>2.37±0.05</td>
<td>+4.53±0.15</td>
<td>0.35±0.23</td>
<td>0.081 exp(-4.54t-1.96t$^2$)</td>
</tr>
<tr>
<td>$k^- $</td>
<td>1.36±0.04</td>
<td>+4.37±0.10</td>
<td>1.00±1.00</td>
<td>0.071 exp(-4.14t-1.30t$^2$)</td>
</tr>
<tr>
<td>$p^- $</td>
<td>5.68±0.13</td>
<td>+6.64±0.15</td>
<td>1.40±0.21</td>
<td>0.077 exp(-4.49t-1.00t$^2$)</td>
</tr>
<tr>
<td>$\pi^- $</td>
<td>2.40±0.06</td>
<td>+5.15±0.18</td>
<td>0.94±0.26</td>
<td>0.082 exp(-4.28t-1.70t$^2$)</td>
</tr>
<tr>
<td>$k^- $</td>
<td>1.50±0.11</td>
<td>+4.96±0.38</td>
<td>1.00±1.00</td>
<td>0.068 exp(-4.29t-1.75t$^2$)</td>
</tr>
<tr>
<td>$p^- $</td>
<td>6.02±0.72</td>
<td>+8.03±0.83</td>
<td>2.70±1.00</td>
<td>0.069 exp(-4.85t-1.29t$^2$)</td>
</tr>
</tbody>
</table>

The values of the parameters are defined from a fit to $M^2 \frac{d^2\sigma}{d\Omega dM^2}$ = $A \exp(Bt+Ct^2)$ measured in the inclusive reaction $ap \rightarrow ax$ at $P_{\text{lab}} = 150-175 \text{ GeV/c}$ in the phase space region $0.05 < |t| < 0.7 \text{ (GeV/c)}^2$ and $0.975 < x_p < 0.995$. 

Table 2

Fermilab Single Arm Spectrometer Group - 1976 [88]
Table 3
Comparison of definitions of triple-Pomeron couplings [96]

<table>
<thead>
<tr>
<th>Authors</th>
<th>Notation used for triple-Pomeron coupling and its relation to $g_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abarbanel et al. [1]</td>
<td>$g_p = g_p$</td>
</tr>
<tr>
<td>Abarbanel et al. [7]*</td>
<td>$r_0 = g_p \sqrt{2}$</td>
</tr>
<tr>
<td>Baker and Ter-Martirosyan [24]</td>
<td>$\sqrt{g_p r} = r_{111} = g_p \sqrt{8 \pi}$</td>
</tr>
<tr>
<td>Capella and Kaplan [53]</td>
<td>$r = g_p$</td>
</tr>
<tr>
<td>Frazer and Moshe [93]**</td>
<td>$r_0 = g_p \sqrt{2}$</td>
</tr>
<tr>
<td>Gribov [109]†</td>
<td>$r_{11} = g_p \sqrt{2}$</td>
</tr>
<tr>
<td>Gribov and Migdal [111]†</td>
<td>$r = g_p \sqrt{2}$</td>
</tr>
<tr>
<td>Kaidalov and Ter-Martirosyan [124]</td>
<td>$r = g_p \sqrt{8 \pi}$</td>
</tr>
<tr>
<td>Migdal et al. [139]</td>
<td>$r = g_p \sqrt{8 \pi} \alpha_0$</td>
</tr>
</tbody>
</table>

* Equation (112) of this reference is incorrect. An additional factor, $\sqrt{2}$, is required on the right-hand side [96]. $r_0$ is taken from $H = 1/2 (\kappa_0 + 1)$ as in eq. (1.11) here.

** Equation 12 of this reference is in error by a factor $\pi \sqrt{2}$.

† Equation (43) is missing a factor of $1/2$.

†† $r$ is defined by fig. 1, and eq. (3) in this reference should read $H = (1/2) \phi \phi + \cdot \cdot \cdot$.

the value of $\ln s_T$ or $\ln s_p$ (namely, $\ln s_p \sim 100$, see also Migdal et al. [139]), it is then clear that much more work has to be done in order to settle this important issue. A possible alternative phenomenological approach to the problem is suggested in the next section (2.5).

2.5. Phenomenological approach to the questions of critical versus non-critical Pomeron and range of validity

The relevance of the critical Pomeron in its perturbative version to the ISR data was studied by Capella and Kaplan [53]. They analyzed the effect of Pomeron interactions on $\sigma_T$ by using a bare Pomeron with an intercept $\alpha_0$. The intercept is treated as a free parameter in order to fit the increase of $\sigma_T$ through the ISR energy range. The total cross section is then given by a power series in the bare triple Pomeron coupling $r_0^2 \ln s$ (fig. 17) and is fitted to the data, fixing the value of $\alpha_0 = \tilde{\alpha}_0$. On the other hand, if one requires that the renormalized Pomeron intercept would be exactly at one (critical Pomeron), such a requirement implies that $\alpha_0 = \alpha_{oc}$ (see sections 1.1 and 1.3.1); namely, the values of the bare parameters in the theory are correlated so that $\alpha_0$ retains a special value $\alpha_{oc}$. The perturbative approach to RFT [53, 54] shows that $\alpha_0 - 1 = \tilde{\alpha}_0 - 1 = \bar{\mu}_0$, which reproduces the rise in the total cross section, is almost order of magnitude bigger than $\alpha_{oc} - 1 = \mu_{oc}$ ($\bar{\mu}_0 = 0.1 > \mu_{oc}$). The inclusion of other Pomeron interactions in the perturbation series does not seem to change this result. If this is the case and $\alpha_0$ is indeed larger than $\alpha_{oc}$, we may end up with a renormalized Pomeron intercept above one and further studies are needed to understand the consistency of the theory with $s$-channel unitarity. This possibility has been intensively studied lately and will be discussed in detail in section 7. The non-critical Pomeron has a considerable appeal from a theoretical viewpoint since it does not require the existence of subtle relations between bare parameters in the Lagrangian as in the case of the critical Pomeron, so that $\alpha_0 = \alpha_{oc}$ in order to give a renormalized Pomeron intercept exactly at one. Since we do not have in RFT any symmetry property such as a gauge principle which will force
\(\alpha(0) - 1 = 0\), the possibility of leaving \(\alpha_0\) free to choose any value, even larger than \(\alpha_{0c}\), and still comply with \(s\)-channel unitarity, is certainly of much interest.

A few more remarks, however, in favor of the critical Pomeron are due. Although the ambiguities in determining \(s_T\) in section 2.4 prevent us from determining \(\log s_T\), there are however many indications that the highest ISR energy region is not ruled out as a possible energy region of phenomenological interest for the scaling results of the critical Pomeron. Several results seem to indicate that the critical scaling behavior might indeed become relevant at energies in the ISR region or possibly only slightly above it. First, there is the interesting prediction [7] for the diffraction peak (fig. 18) which has the features of the ISR data. There is also a possibility to understand the features of the central plateau through the critical Pomeron when the approach to scaling is taken into account [141]. The value of \(\mu_{0c} = 0.01\), calculated by Capella and Kaplan [53] in perturbation theory and found to be too small by an order of magnitude to produce the data, was calculated nonperturbatively by Frazer et al. [95] and found to be larger by a factor 5–7 than the value obtained by Capella and Kaplan. One may argue that the nonperturbative calculation of \(\mu_{0c}\) is more adequate in the case of a renormalized Pomeron with an intercept at one. In fact, this is the only consistent way [157] enabling us to calculate \(\mu_{0c}\) if no \(k^2\) dependence is introduced into the triple Pomeron coupling \(r_0\), as used in [53]. Corrections at finite energies to the AGK cutting rules (section 3.2) due to energy momentum conservation [56] had also shown that the critical Pomeron is not ruled out. Finally, in view of the difficulties explained in section 2.4 in determining \(s_T\), one cannot rely on the value in eq. (2.48) but of course this value is not ruled out as an attractive possibility.

In conclusion, it seems that further clarification of the issue of a critical versus non-critical Pomeron, and much insight into the question of the range of validity of RFT may be obtained in the future by a thorough phenomenological study of the problem. We believe that a phenomenological approach, not based on perturbation theory à la Capella and Kaplan [53], but on the scaling solutions with the critical Pomeron, would be a promising method for reaching these goals. In such a treatment, the scaling solution, as well as the approach to scaling, will be confronted with the elastic and inelastic scattering data in the ISR energy region. The main effort in such a study will be the appropriate evaluation of the terms which govern the approach to scaling, such as in eqs. (2.23) and (2.39), as well as contributions from other Pomeron interactions. The non-universal coefficients extracted from different sets of measured cross sections can be related to each other and a consistent fit, hopefully with a small number of non-leading terms, will tell us how close we are to the scaling region, if at all (see ref. [11P] in the Note added in proof). If this suggested program will fail, we will be left with the only other possibility, namely \(\alpha_0 > \alpha_{0c}\). (This case will be discussed in section 7.)

A quite different approach to the whole issue was advocated by several authors [81], triggered mainly by the phenomenological studies of the Pomeron's intercept above one [53, 54, 128]. It was suggested to formulate the theory at high \(s\) in terms of a certain type of singularity, the “Froissaron” \((-\log s)^2\), or the “superpropagator” that is simply the result of the eikonalization of multi Pomeron \((s^3)\) exchange (fig. 6b). Gribov calculus (section 1) can then be used to analyze graphs with such “Froissaron” exchanges and interactions. It was argued that \(s\)-channel and \(t\)-channel unitarity constraints are simultaneously satisfied in such a theory. The resultant elastic amplitude corresponds to a black disc with radius \(B ~ \log s\). It is difficult to see a clear relation between this approach and the one described in section 7 since two different types of graph summations are involved.
3. Production amplitudes and analysis of the s-channel content of Reggeon field theory

One of the major issues and recently a very active research area is the analysis of the s-channel content of Reggeon field theory. As explained in the Introduction, Reggeon field theory satisfies, by construction, t-channel unitarity. There is presently no rigorous proof that s-channel unitarity relations are satisfied in RFT. A possible approach, however, to this problem is to investigate whether particular aspects of rigorous requirements imposed by s-channel unitarity are satisfied in RFT.

The analysis of the j-plane structure of production amplitudes and their asymptotic behavior sheds some light on this very important problem. An interesting issue is studying the answer given by the strong coupling scaling solution to two of the major problems found in the weak coupling solution with a Pomeron intercept at one: the Finkelstein–Kajantie problem [90], and the decoupling problem [37]. The results discussed below show that these difficulties do not appear in RFT (Migdal et al. [139], Abarbanel et al. [7], Cardy [59], Frazer and Moshe [93], Bartels and Rabinovici [27]), and therefore are very encouraging. Another step towards understanding the s-channel content of the Pomeron was made by Abramowskii, Gribov and Kancheli [13], and formulated in a cut Reggeon field theory by Suranyi [158], Cardy and Suranyi [61], and Ciafaloni, Marchesini and Veneziano [67]. Other aspects of the s-channel issues for the critical Pomeron ($\alpha_0 = \alpha_{0c}$), and for the sub-critical Pomeron ($\alpha_0 > \alpha_{0c}$), will be discussed in sections 6 and 7.

3.1. Solution to the decoupling problem and the remedy for the Finkelstein–Kajantie disease

The analysis of the triple Pomeron region in theories where the Pomeron singularity is asymptotically a simple moving pole with an intercept at one implies that the triple Pomeron coupling will vanish at zero momentum transfer [1, 77]. However, not only $g_{ppp}(t_i = 0)$ does not approach zero experimentally, but as mentioned in section 1, it was also shown that such a Pomeron will decouple from elastic processes (Jones et al. [122], Brower and Weis [37]) and cuts have to play the crucial role in the leading j-plane singularity. In Reggeon field theory we start with a bare Pomeron pole $\alpha(t) = \alpha_0 + \alpha_0 t$, and the effect of the cuts is introduced by treating the multi-Pomeron exchange through the field theory Lagrangian in eqs. (1.9) and (1.11). As a result, the full renormalized Pomeron has a complicated cut structure in the complex angular momentum plane and one may ask whether this Pomeron singularity is not suffering from similar disastrous consequences as the Finkelstein–Kajantie disease or the decoupling theorems.

In general, the analysis of production amplitudes is more complicated than the study of elastic amplitudes since, in addition to complex angular momentum, complex helicity has to be also introduced as conjugate variables to the Toller angle variables (Jones et al. [121]). The triple Regge limit of the single particle inclusive cross section, which is an helicity pole limit [76], is considerably simplified, being dominated by a single helicity amplitude (the maximum helicity flip amplitude [3, 102]). The remaining helicity amplitudes are down by powers of $s/M^2$ ($s/M^2 \gg 1$ in the triple Regge region).

The rules of the calculations in the triple Pomeron region were first derived by using Reggeon unitarity relations [58]. They derived [7] from a study of hybrid Feynman graphs of the type shown in fig. 20 for the six point function and also formulated by using path integrals [142] (section 5.3). The new ingredient that exists here has to do with the fact that the inclusive cross section is a function of the variables $Y = \log s$ and also $\eta = \tau = \log M^2$ (fig. 21). The amplitude therefore is also a function of an intermediate “time” η, and holding an intermediate time fixed implies an “energy” ($E = 1 - j$)
non-conserving process. Any given diagram has a number of "branching vertices", $r_B$, at which the energy is not conserved. The new vertex $r_B$ is taken to be identical to $r_0$ since a $k^2$ dependence in it will not change the infrared behavior. This also implies that $r_B$ is unchanged when the appropriate $M^2$ discontinuity is taken. (We will not discuss here other subtle problems involved in taking this discontinuity; see, for example, refs. [149 and 140].)

The leading contribution to the inclusive cross section in the triple Pomeron region comes from the
diagram in fig. 22a. $\hat{F}^{1,2}(E_1, k_1)$, the energy non-conserving single particle irreducible proper vertex, was
calculated using the method described in sections 2.1 and 2.3. The first order in $\epsilon$ calculation gives [93]):

$$M^2 \frac{d\sigma}{dM^2 dt} = \frac{N(0)N^2(t)}{16\pi} \sqrt{2r_0} \Phi(\frac{1}{2}; -\kappa \log M^2)$$

$$\times \left\{ \sum_n \frac{\alpha n^2 \log(s/M^2)^n}{n!} \Phi \left( \frac{1}{6} - \frac{n + 1}{12}, n + 1; -\kappa \log(s/M^2) \right) \right\}^2.$$  \hfill (3.1)

Again, here $\Phi(a, b; z) = F_{1,2}(a, b; z)$, and again, in terms of the renormalized parameters at small values of the scaling variable $t[\ln(1/(1-x))]^{1-\epsilon}$, one finds, as in the elastic scattering, a leading exponential diffractive peak.

---

Fig. 22. The leading (a) and non leading (b,c) contributions to the inclusive cross section in the triple Pomeron region.
\[
\frac{d\sigma}{dt \, dx} = \frac{N_R(0)N_R(t)}{16\pi} \sqrt{2r} (E_N \log M^2)^{-\epsilon/4 + (1/2)\epsilon - \xi} \\
\times \left[ -E_N \log(1-x) \right]^{-2\gamma}
\cdot \frac{1}{1-x} \exp \left\{ \frac{2\alpha_s}{E_N} \left[ -E_N \log(1-x) \right]^{-\epsilon} \right\}.
\]

(3.2)

Here \( x \) is the Feynman \( x = 1 - M^2/s \).

The method developed by Abarbanel et al. [7] is applicable at higher values of the scaling variable and the calculated diffraction peak shows here at \( \ln M^2 \gg \ln(s/M^2) \gg 1 \) a structure similar to that of the elastic diffraction peak. “Asymptopia” is determined here by the subenergies \( M^2, s/M^2 \) and even at the ISR these energies are probably too small to show the predicted forward diffraction structure. The approach to this asymptotic result is governed by terms of order \( (\log M^2)^{-\lambda} \) and \( \log(1-x)^{-\lambda} \) and possibly from “non enhanced” graphs [93] (figs. 22b,c). The contribution of eq. (3.2) to the total cross section is:

\[
\sigma_T^{\text{3P}} \sim (\log s)^{-(3/2)\gamma - (D/4)(1-\epsilon)}
\]

(3.3)

which has in the \( \epsilon \) expansion a lower power of \( \log s \) than the asymptotic total cross section, namely:

\[
\frac{\sigma_T^{\text{3P}}}{\sigma_T} \sim (\log s)^{-1+\epsilon/4} \xrightarrow{s \to \infty} 0.
\]

(3.4)

Note that in general from eqs. (3.3) and (2.39) one finds

\[
\sigma_T^{\text{3P}}/\sigma_T \sim \ln s^{(1/2)(-\gamma - (D/2)(1-\epsilon))} \sim (\sigma_{\epsilon l}/\sigma_T)^{1/2}
\]

(3.5)

which is, if \( \sigma_{\epsilon l} < \sigma_T \) is satisfied in RFT, then

\[
-\gamma - \frac{1}{2}D(1 - \epsilon) \leq 0
\]

(3.6)

and \( \sigma_T^{\text{3P}} < \sigma_T \). As mentioned in section 2, studies of Reggeon field theory on a lattice (Cardy and Sugar [57]) show that the inequality (3.6) is indeed satisfied at \( D = 2 \) (see section 6). Thus, Pomeron interaction in eq. (1.11) succeeded in preventing the difficulty encountered in theories with weakly coupled Regge cuts. There is no need for \( g_{\text{PPP}}(0) = 0 \) \( (M^2 \, d\sigma/dt \, dM^2 \neq 0 \) at \( t = 0 \), and consequently no decoupling problems appear in RFT in the case \( \alpha_0 = \alpha_{\text{oc}} \).

Multiparticle production amplitudes in Reggeon field theory are another source for the study of its \( s \)-channel content. The \( n \) particles production amplitude \( T_{2-n} \) cannot in general be written as a multiple Sommerfeld–Watson representation of a single amplitude. The amplitude is decomposed into different terms, each representing a set of allowed simultaneous discontinuities, consistent with the Steinmann [154] relations. (For a review see Brower, DeTar and Weis [37], and White [168,]). The structures of \( T_{2-3} \) and \( T_{2-4} \), which can be derived from analyticity and unitarity alone, were generalized by Bartels [26] to \( T_{2-n} \) showing a physical region factorization in the multi Regge region, similar to \( T_{2-3} \) and \( T_{2-4} \). Also, the conditions for this factorization can be proven directly using unitarity and analyticity only [168]. The general structure of the \( T_{2-n} \) amplitude is simplified in the case in which only Pomerons are exchanged. It degenerates into a single amplitude of the form

\[
T_{2-n}(y_1 \ldots y_{n-1}k_1 \ldots k_{n-1}) = \frac{(i)^{n-1}s}{(2\pi i)^{n-1}} \int dE_1 \ldots dE_{n-1} e^{-y_1 E_1} \ldots e^{-y_{n-1} E_{n-1}} F_n(E_1 \ldots E_{n-1}, k_1 \ldots k_{n-1})
\]

(3.7)

where \( k_i^2 = -t \) and \( y_i \) are the rapidity gaps between the produced particles (fig. 23). The singularity structure in the Toller angles appears in the complex function \( F_n \) through its dependence on \( k_i \cdot k_j \).
The RFT Feynman rules for the calculation of $F_n(E_k)$ were defined by Bartels [26]. The new ingredient is a complex vertex $V_0$, describing Pomeron–particle–Pomeron vertices, which adds a term, $V_0 \bar{\psi} \psi$ to the Lagrangian. It was shown [27], however, that the result of such a calculation of the scaling behavior will be inconvenient for performing the phase space integration of $T_n^* T_n$ to obtain $\sigma_n$. Bartels and Rabinovici [27] have therefore proceeded differently by formulating a method for calculating $\sigma_n$ directly. The difficulties one finds from divergences in the phase space integration when constants $r_0$ and $V_0$ are used, turn in their method, into the usual divergences in the field theory which can be regularized and renormalized. The essential ingredient of their method is graphically shown in fig. 24. The quantity $\sigma_n(E)$ then calculated, is the Mellin transform of $\sigma_n(s)$. If the new real four Pomeron coupling is defined as $U_0 = V_0 V_0^*$, then $\sigma_n(E)$ can be calculated using the same RFT rules with which the elastic amplitude was calculated. Two Pomeron fields, $\psi_+$ and $\psi_-$, have to be defined in order to represent the Pomerons above and below the s-channel cut in fig. 24, respectively (see also section 3.3). The Lagrangian is

$$\mathcal{L} = \sum_{i=+,-} \left[ \mathcal{L}_0 - \frac{1}{2} \psi_i(\bar{\psi}_i + \psi_i) \psi_i \right] - U_0 \bar{\psi}_+ \psi_+ \psi_- + J \bar{\psi}_+ \psi_+ + J^+ \bar{\psi}_- \psi_- + \ldots$$

(3.8)

and $\sigma_n(E)$ is proportional to $U^{n-1}$.

In the presence of the cuts introduced by the triple Pomeron interaction, the new effective four Pomeron coupling $U$ is infrared free (at least in the $\epsilon$ expansion) at $D = 2$ [27]. Namely, the $\beta$ function associated with $U$ is $\beta_U = a U + O(U^2)$, where $a > 0$. The asymptotic dependence of $\sigma_n(E)$ on $E$ is independent of $U$ and therefore $\sigma_n(E)$ has the same critical exponents as $\sigma_{el}(E)$. The approach to scaling in $\sigma_n(E)$ is however determined by $a$ and therefore:

$$\sigma_{n+2}(E) \sim U^n E^{2\gamma - 1 + z(D/2)} \sum_{k=1}^{n+1} C_{n+k} E^{a(n-k)}.$$  

(3.9)

The different contributions to $\sigma_n$ in eq. (3.9) can be understood as follows: The effective coupling $U(-t)$ (not to be confused with the physical one, $U$ in eq. (3.9)), behaves like $U(-t) \sim U e^{at}$ and vanishes in the infrared region ($t \to -\infty$). The leading contribution $\sigma_n(s) \sim U^n \sigma_{el}(s)$ comes from a phase
space region, where there is only one large rapidity gap between the produced particles. For each rapidity gap a physical $U$ is replaced by an effective $U(-t) \sim U e^{t}$, thus multiplying $\sigma_{el}(E)$ by a power of $E^{a}$ and therefore the configuration of $k$ large gaps gives $\sigma_{n}(s) \sim U^{n} \sigma_{el}(s) \left( \ln s \right)^{-ak}$. The configuration of $n$ large gaps which contributes to $\sigma_{n}$ a term $\sigma_{0}(\ln s)^{-an}$ was wrongly considered by Migdal et al. [139] to be the leading part.

Using $\sigma_{n}(s)$ (from eq. (3.9)) one finds, summing all leading terms for small $U$ that $\Sigma \sigma_{n} \sim \sigma_{el}$. If $U$ is large, the series $\Sigma \sigma_{n}$ diverges when only leading terms are taken. Such a situation forces the inclusion of the non-leading terms of $\sigma_{n}$ in calculating the sum and one has to check whether the new built-up power of $s$ does not violate the Froissart bound, as it does in the Finkelstein–Kajantie case [90]. Studying the form of $\beta_{U}$ assures us indeed the existence of a range of values of $U$ in which the effective $U(t)$ is driven to zero in the infrared limit and no new singularity to the right of $j = 1$ is developed. Near $D = 4$, where the effective triple Pomeron coupling is small, the calculation of $\beta_{U}$ to order of $U^{2}$ gives a value for the upper limit $U < U_{c}$, for which $U(t) \rightarrow 0$ in the infrared region. At $D = 2, a > 0$, and nothing is known about the large $U$ structure of $\beta_{U}$ [27].

The conclusion, therefore, is that as long as the physical Pomeron–Pomeron–particle vertex is not too large, there will be no violation of $s$-channel unitarity in production processes in RFT since $\Sigma \sigma_{n} \sim \sigma_{el}$. In section 6 we will prove, however, that RFT results indeed satisfy $\sigma_{el} < \sigma_{T}$; this means that unlike self-consistent models (Finkelstein and Zachariasen [91]) in RFT, the production processes with repeated Pomeron exchange do not build the leading behavior of $\sigma_{tot}$ [27] and contributions from production processes with secondary trajectories probably play here a major role.

Diffractive production amplitudes were studied in RFT [137] also for the case $\alpha_{0} > \alpha_{0c}$, where, again, no conflict with $s$-channel unitarity constraints was found. This will be discussed in section 7.

### 3.2. Abramovskii, Gribov and Kancheli (AGK) cutting rules

Studying the $s$-channel content of RFT, one also faces a basic question involving the bare Pomeron structure: what is the $s$-channel content of the bare Pomeron in terms of particle production amplitudes? Since a detailed scheme for building the Pomeron out of an underlying strong interaction theory is not yet available, the answer to this question has to rely of course on a few assumptions. One would like to see the gross features of high energy production data to be revealed by the structure of the absorptive part of two body amplitudes in its first approximation, namely, in a single Pomeron exchange. The absorptive part of the bare Pomeron, given in terms of production amplitudes should reproduce a uniform spectrum in rapidity for the inclusive cross section in the central region, and a sharp decrease in transverse momentum. The multiperipheral model has these qualities and it is therefore common to assume that the absorptive part of the bare Pomeron can be approximately represented by a generalized ladder (fig. 25a), without specifying the character of the exchanges in the production amplitude (Abramovskii et al. [13]). It is assumed, however, that states with large four momentum squared on virtual lines are suppressed. More complicated interacting Pomeron graphs can be also “cut” to give contributions to the production amplitude as shown in fig. 25b. At very high energies, when contributions of the type shown in fig. 6c to the elastic amplitude become important, it is realized that the cutting of these very many diagrams and their summation is far from being trivial (for a detailed review see [24]). A practical way was suggested to keep the counting of diagrams straight and to evaluate their approximate sum by employing Reggeon field theory for cut and uncut Pomerons (see section 3.4).

The assumption stating that large $q^{2}$ are suppressed on virtual line implies that only cuts of the type
of figs. 25a and 25b are important. Cuts that leave part of the Pomeron uncut, result in a large $q^2$ flowing in the peripheral chain. Ignoring for a moment Pomeron interactions, the amplitude of fig. 6b can be written as

$$iA^{(n)}(s, t = -Q^2) = s \int N_n[(iP_1)(iP_2) \ldots (iP_n)]N_n \, d\Omega_n$$

(3.10a)

where

$$d\Omega_n = \frac{1}{n!} \delta^2 \left( Q - \sum_{i=1}^{n} k_i \right) \prod_{i=1}^{n} \frac{d^2 k_i}{2(2\pi)^2}$$

(3.10b)

and

$$P(s, t = -k^2) = s^{\alpha(t)-1} \frac{\exp\{-i\frac{1}{2} \pi \alpha(t)\}}{\sin\{\frac{1}{2} \pi \alpha(t)\}}$$

(3.10c)

The analysis based on hybrid Feynman diagram models for the $n$ Reggeon cut [13] shows that cuts through $N_n$ (the coupling of $n$ Reggeons to a particle line) leave it unchanged, essentially because the uncut $N_n$ can be written as an integral over absorptive part of $k$ Reggeon-particle → $n-k$ Reggeon-particle amplitude, and this is identical to the cut $N_n$. The couplings $N_n$ are non-planar diagrams; a simple $\phi^3$ diagram for $N_2$ with non-vanishing third spectral functions is shown in fig. 26. An analysis within a $\phi^3$ theory of the allowed $s$-channel intermediate states in the unitarity equations of these diagrams was carried out in [114].

Calculating the $s$-channel absorptive part of $A^{(n)}$ is now a matter of keeping track of the different combinations. Denote

$$2 \text{Im } A^{(n)} = \sum_{k=0}^{n} F_k^{(n)}$$

(3.11)

where the summation is on the number ($k$) of cut Pomeron and
\[ F_{k}^{(n)} = (-1)^{n-k} \left( \binom{n}{k} \prod_{j=1}^{n} (2 \text{Im } P_j) + \delta_{k,0,2} \text{Im } \left( -i \prod_{j=1}^{n} (iP_j) \right) \right). \] (3.12)

Since \( \Sigma_{k=0}^{n} (-1)^{k} \binom{n}{k} = 0 \), we have the simple result

\[ 2 \text{Im } A^{(n)} = 2 \text{Im } \left( -i \prod_{j=1}^{n} iP_j \right). \] (3.13)

Note that since also \( \Sigma_{k=1}^{n} (-1)^{k} \binom{n}{k} = 0 \), the contribution of the \( n \) Pomeron cut to single particle inclusive cross section in the central region vanishes (figs. 27a and 27b are graphical representations of the two simplest examples in the case \( n = 2 \)). The same is true for \( r \) particles inclusive cross section since \( \Sigma_{k=1}^{n} (-1)^{k}(k-1) \ldots (k-r+1)(k) = 0 \). All this has of course an important impact on determining the leading contributions for inclusive cross sections at asymptotic energies.

The question of whether the above AGK assumption on the property of \( N_\omega \) is true in general has been, however, the subject of some controversy (DeGrand [73], DeGrand and DeTar [74], Koplik and Mueller [127], McLerran and Weis [138]). In general, the AGK counting rule for fig. 27a is replaced by [73], \( 2P - 4(P + xP) + 4xP = -2P \), and \( x \) depends crucially on the analytic properties of \( N_\omega \). In order to compare the cuts through one or both Reggeons with the diffractive cut (fig. 27a) it is necessary to take discontinuities involving overlapping-channel invariants in the two Reggeon slice (fig. 27a-III). It was shown (Steinmann [154], Cahill and Stapp [47]) that double discontinuities in overlapping kinematic variables vanish in the physical region of those variables. This is the case for the Mandelstam graph (fig. 26). However, restoration of the AGK rules occurs here due to the presence of anomalous singularities in the complex plane, and one may ask whether this interesting compensation is not unique to the Mandelstam graph. (For other implications of anomalous singularities in \( 3 \to 3 \) amplitudes see also Patrascioiu [149] and Moshe and Patrascioiu [140].) In fact, Reggeon-particle coupling, which possesses only normal thresholds [73], results in a counting \( (x = 0) 2P - 4P + 0 = -2P \) in fig. 27a.

The analytic structure of \( N_\omega \) was studied by DeGrand and DeTar [74] in a simple dual model. The Reggeon particle amplitude obtained from the six point dual amplitude is quite different from the Feynman graph model. In particular, the double Regge cut diagram (fig. 27a-III) contains no poles in an overlapping variable. If the contour of integration can be closed at infinity, then this contribution vanishes [74]. McLerran and Weis [138] have shown however that although the analytic structure of the dual model and Feynman graph model (e.g. Mandelstam graph) is different, the cut through both Reggeons in fig. 27a-III does not vanish in any of these cases. Indeed, the reason for non-vanishing is different in each case. In the Feynman graph, as mentioned above, the non-vanishing piece is due to higher order Landau singularities [114]. In the dual model one finds that the contour of integration cannot be closed at infinity due to the existence of an essential singularity at infinity and therefore the

\[
\begin{array}{ccc}
\text{(a)} & \text{(b)}
\end{array}
\]

Fig. 27. (a) The AGK counting rules for the cutting of the two-Pomeron exchange diagram. (b) The vanishing sum of the contribution of the two Pomeron exchange to the inclusive cross section in the central region as obtained from the AGK rules. (The factor 2 in the second diagram comes from the fact that the detected particle can come either from the left or right cut Pomeron.)
slice through both Reggeons does not vanish and the AGK weights of the various discontinuities are exactly reproduced. In general however for \( \log s \leq (Rm)^2 \) dual models have no simple cutting rules. In case of particle nucleus scattering \( R \) in the nucleus radius \( (m \sim O(1 \text{ GeV})) \) and therefore, for large nuclei we might not be able to employ the AGK rules. In the planar dual string model (for a review see [136]) the AGK rules are not satisfied and a weight with \( x = 0 \) is obtained [127]. If, however, the different ends of the string scatter off different sources then such a non-planar graph satisfies the AGK rules. These cutting rules are also well supported in the framework of soft field theory [127].

A different approach to the problem by Cardy and Suranyi [61] produces a very interesting confirmation of the AGK rules by imposing \( s \)-channel unitarity requirements on the Reggeon field theory Lagrangian; thus, deriving the AGK cutting rules from an \( s \)-channel condition imposed in a \( t \)-channel unitary theory. This will be discussed in detail in section 3.4. Imposing unitarity to all orders in \( 1/N \) in the topological expansion [67] had been another convincing confirmation of the AGK rules.

Bearing in mind the fact that the AGK cutting rules were derived, as seen above, within several different frameworks, we are tempted to accept their general status.

A slight generalization of these rules [13] enables us to use them in arbitrary Reggeon diagrams. A general Reggeon graph can be written in a form similar to eq. (3.10a), including however products of \( P_t \)'s and vertex functions. The vertex \( \Gamma^{1,n} \) is then unchanged by cutting it, for the same reason \( N_n \) is unchanged, but \( \Gamma^{m,n} \) with \( m, n > 1 \) is in general changed when cut. One has to know the values of cut and uncut vertices in order to continue from here. If we restrict ourselves to triple Pomeron interactions only, there are no further complications since \( r \) is unchanged by cutting.

### 3.3. The inclusive cross section in the central region

The calculation of the asymptotic distribution in the central region demonstrates a simple application of the AGK rules discussed in section 3.2. This will be discussed here; another application for particle nucleus inclusive reactions (Schwimmer [153], Kancheli [125]) will be discussed in section 5.1. The incorporation of these rules in cut Reggeon field theory will be presented in section 3.4.

Isolating the renormalized two cut-Pomerons–two particles vertex, \( \hat{\Gamma}^2 \) in Reggeon graphs for the six point function, and applying the AGK rules (as, for example, in fig. 27b) shows that these contributions to the asymptotic inclusive cross section vanish. The non-vanishing contributions come from Reggeon diagrams (fig. 28) in which the two cut-Pomerons–two particles vertices, \( \hat{\Gamma}^2 \), is unrenormalized by multi Pomeron exchanges and interactions (Migdal et al. [139], Caneschi and Jengo [50], Suranyi [158]). The cross section is therefore dominated by the graph in fig. 28, where two renormalized Pomeron propagators \( G_{r}^{1,n} \) are attached to the unrenormalized \( \hat{\Gamma}_2 \) and similar graphs with \( r + 1 \) renormalized propagators will dominate the \( r \) particle inclusive cross section.

Using the scaling behavior of \( \Gamma^{1,1} \) in the strong coupling solution (eq. (1.20)) one finds:

\[
\langle n \rangle \sim (\log s)^{1-\gamma}, \quad \langle n^k \rangle \sim (\log s)^{k(1-\gamma)}
\]  

(3.14)
rising faster than in the multiperipheral model \((-\gamma > 0)\). The ratio \( c_k = \langle n^k \rangle / \langle n \rangle^k \) can be calculated in

![Fig. 28. The leading contribution to the inclusive cross section in the central region in Reggeon field theory.](image)
an $\epsilon$ expansion; it is asymptotically a constant, as seen from eq. (3.14). The departure of $c_k$ from one indicates the existence of long range correlations due to multi cut Pomeron exchange [50, 158].

The approach to scaling discussed in section 2.2 is of particular interest for the calculation of the inclusive cross section in the central region, since the subenergies involved are less high ($y < Y$) and the non-leading terms have then a significant contribution to the cross section. The expression for the inclusive cross section in the central rapidity region, including the terms that govern the approach to scaling, is [141]:

$$
\frac{d\sigma}{d^2\mathbf{p}_T dy} = f(P_7^2) \left( \frac{Y}{2} - y \right)^{-\gamma} \left( \frac{Y}{2} + y \right)^{-\gamma} \left\{ 1 + A \left[ \left( \frac{Y}{2} - y \right)^{-\lambda} + \left( \frac{Y}{2} + y \right)^{-\lambda} \right] + A^2 \left( \frac{Y}{2} - y \right)^{-\lambda} \left( \frac{Y}{2} + y \right)^{-\lambda} + B \left[ \left( \frac{Y}{2} - y \right)^{-2\lambda} + \left( \frac{Y}{2} + y \right)^{-2\lambda} \right] + O \left[ \left( \frac{Y}{2} \pm y \right)^{-3\lambda} \right] \right\}.
$$

(3.15)

$f(P_7^2)$ is an exponential in $-P_7^2$, and is proportional to the unrenormalized two Pomeron–two particles vertex $\hat{F}_0^2$. $Y = \log s$ is the total rapidity, and $y$ is the rapidity of the detected particle. The asymptotic centrally peaked distribution $\sim (Y/2 - y)^{-\gamma}(Y/2 + y)^{-\gamma}$ agrees, of course, with Migdal et al. [139], and is due to the presence of the two $G_{R,1}$ in fig. 28. The “non-universal” coefficients $A$ and $B$ were evaluated in the first loop approximation and their general form in higher loop calculation is also known (section 2.2 and ref. [141]). Written in terms of unrenormalized parameters in the first loop calculation the constants $A$ and $B$ can be estimated from the strength of the measured triple Pomeron coupling $\alpha_0$. Being “non-universal”, ultimate values of $A$ and $B$ are determined by comparison with experiment, as has been done for the first order $\epsilon$ calculation. There are additional non-leading terms coming from replacing $G_{R,1}$ in fig. 28 by $G_{R,1}^{1,2}$. These “non-enhanced” graphs contribute terms of order $(Y/2 \pm y)^{-\lambda}$, compared to $(Y/2 \pm y)^{-\gamma-\lambda}$, which are the first two non-leading terms from $G_{R,1}$. If the present “world averages” evaluations of $-\gamma$ and $\lambda$ (table 1) are close to their true values, then the non-enhanced contributions are less important than the three non-leading terms presented in eq. (3.15); otherwise, they should be added there. It is, however, difficult to decide whether the non-leading contributions of “irrelevant” parameters in Wilson’s sense $\mathscr{L}_1 \sim \lambda_{10}(\bar{\psi}^2\psi^2 + \lambda_{20}\bar{\psi}^4)$ might be of comparable size to the first non-leading term $(\log s)^{-\gamma-\lambda}$.

It is interesting to note the effect of the non-leading terms in eq. (3.15) on the direction of approach to the centrally peaked [141] $d\sigma/dy \sim (Y/2 - y)^{-\gamma}(Y/2 + y)^{-\gamma}$ asymptotic distribution. For $A > 0$, the leading correction to scaling in eq. (3.15) gives a term which enhances the distribution away from the center; so there will be a smooth transition from an almost “plateau”-like distribution at high energies (ISR – if eq. (3.15) is applicable) to a centrally peaked distribution at higher energies. $A$ is positive in the first loop calculations and may preserve its sign at higher orders.

To end with an optimistic remark, accompanied however with a word of caution, we will look at the $y$ and $Y = \log s$ dependence of $d\sigma/dy$ in eq. (3.15). Using for $-\gamma \approx 0.35 - 0.5$, $\lambda \approx 0.3 - 0.5$ and fitting $A$ and $B$, two features of the ISR data [16] are reproduced [143]: (a) the 10–15% slope of $d\sigma/dy$ as a function of $y$; (b) the 30–40% rise with $s$, within the ISR energies of the central “plateau”. However, the fact that eq. (3.15) seems to describe the data, as early as the ISR energies, is somewhat surprising because the rise of the central “plateau” in eq. (3.15) is due to the rise of the Pomeron enhanced graph in a smaller subenergy than the energy at which $\sigma_T$ starts to rise. The measured strength of the short range correlation does not seem to support the possibility that the lower Regge trajectories couple with relatively weaker strength in $d\sigma/dy$ than in $\sigma_T$. The possibility of a surprisingly early scaling in the inclusive cross section is reminiscent of the surprisingly early agreement with the data of the elastic differential cross section discussed in section 2.3 (see fig. 18). It is very desirable to try to
understand the early appearance of the asymptotic results; the other alternative that remains, of
course, is to regard it as a mere coincidence.

Other schemes for treating the inclusive cross section [68, 142], which may be more relevant to
present energies, will be discussed in section 5.

3.4. Cut Reggeon field theory

Even with the simplifications introduced by the AGK rules in section 3.2 keeping track of all
possible s-channel cuts in a complicated diagram may be very tedious. The way to incorporate these
rules in RFT is to describe a cut diagram by three different Pomeron fields appearing now. \( \psi_+ \) and \( \psi_- \)
are fields associated with the Pomerons above and below the s-channel cut, respectively, and \( \psi_c \), the
field associated with a cut Pomeron. Some of the allowed interactions between these three fields are
shown in fig. 29. In general, \( \psi_+ \) and \( \psi_- \) can be thought of as representing Reggeons with regular and
complex conjugate signature factors, while \( \psi_c \) represents a Reggeon whose signature factor is replaced
by its imaginary part [158, 61, 67].

The free Lagrangian is given by

\[
\mathcal{L} = \frac{1}{2} \bar{\psi}_c \frac{\partial}{\partial \tau} \psi_c + \frac{1}{2} \bar{\psi}_+ \frac{\partial}{\partial \tau} \psi_+ + \frac{1}{2} \bar{\psi}_- \frac{\partial}{\partial \tau} \psi_-
- \Delta(\bar{\psi}_c \psi_c + \bar{\psi}_+ \psi_+ + \bar{\psi}_- \psi_-) + \alpha(\nabla \bar{\psi}_c \nabla \psi_c + \nabla \bar{\psi}_+ \nabla \psi_+ + \nabla \bar{\psi}_- \nabla \psi_-).
\]

This is the usual RFT free Lagrangian (eq. (1.9)) for each of the fields \( \psi_+ \), \( \psi_- \) and \( \psi_c \). In the interaction
Lagrangian, certain selection rules have to be imposed to ensure that no disconnected diagrams
appear on either side of the cut. For example, there is no \( \psi_- \) and \( \psi_+ \) interaction without involving a \( \psi_c \);
the decay of \( \psi_c \) into two \( \psi_+ \) or two \( \psi_- \) is not allowed, etc. The following interaction Lagrangian can be
shown to generate all possible cuts of all possible diagrams in a Reggeon field theory with triple
Pomeron interactions:

\[
\mathcal{L}_{int} = -\frac{1}{2} G(\bar{\psi}_c \psi_c^2 + \text{h.c.}) - \frac{1}{2} G^*(\bar{\psi}_c \psi_c^2 + \text{h.c.}) - \frac{1}{2} g_0(\bar{\psi}_c \psi_c^2 + \text{h.c.})
- g_1(\bar{\psi}_c \psi_c \psi_+ + \text{h.c.}) - g_1(\bar{\psi}_c \psi_c \psi_+ + \text{h.c.}) - g_2^*(\bar{\psi}_c \psi_c \psi_- + \text{h.c.}).
\]

The couplings \( G, g_0, g_1 \) and \( g_2 \) are, for the moment, independent. The work of Abramovskii et al. [13]
discussed in section 3.2 was based on a study of the analytic properties of a model of Feynman
diagrams that describes the triple Pomeron vertex and \( N_n \). They then assumed that these properties
are true in general and obtained relations between \( G, g_0, g_1 \) and \( g_2 \) being one vertex cut in different
ways. This, however, as explained in section 3.2, was subject to controversy; the different approach
of Cardy and Suranyi [61] sheds much light on the issue while strengthening the AGK results. Their

![Fig. 29. The absorptive part of a multi Pomeron exchange diagram with possible interactions between the \( \psi_+ \), \( \psi_- \) and cut Pomeron field \( \psi_c \).](image-url)
approach is based on imposing the Lagrangian in eq. (3.17) conditions of s-channel unitarity. Taking into account that the Pomeron propagator is purely imaginary they required that the \( \langle \psi_0 \bar{\psi}_0 \rangle \) propagator would equal to \( \langle \psi_i \bar{\psi}_i \rangle \) equating the high energy behavior of the amplitude and its s-channel discontinuity to all orders in perturbation theory. These imposed equalities give rise to three possible solutions for the ratios between the coupling constants. Only one solution gives also the right value for the diffractive contribution to the two Pomeron cut discontinuity

\[
g_0 = 2g_1 = \pm i\sqrt{2}G = \pm i\sqrt{2}g_2. \tag{3.18}
\]

These are also the ratios between the couplings that agree with Abramovskii, Gribov and Kancheli [13]. The relations in eq. (3.18) imply also proportionality relations for \( 1 \rightarrow n \) Reggeon amplitudes as in the AKG rules. This does not hold for \( n \rightarrow m \) amplitudes if \( n, m > 1 \), which clarifies why the Lagrangian in eqs. (3.16) and (3.17) cannot be fully diagonalized. A partial diagonalization is however very helpful in practical calculation (see e.g. ref. [142] and section 5.3).

Cardy and Suranyi’s approach [61] to the problem is of much interest to the issue of the s-channel content of Reggeon field theory. Bearing in mind the general status of the AGK cutting rules (section 3.2) it is reassuring to find out that they can be rederived in RFT by imposing s-channel unitarity requirements on the theory, thus incorporating s-channel constraints in our t-channel unitary theory.

The cut Reggeon field theory Lagrangian in eqs. (3.16–17) with the relations in eq. (3.18) was derived also from the topological expansion when unitarity had been imposed in all orders in the \( 1/N \) expansion [67].

Further implications of eqs. (3.16)–(3.18) on the s-channel constraints in RFT were studied by Ciafaloni and Marchesini [68] in the central rapidity region of the inclusive cross section and by Moshe and Paige [142] in hadron hadron and hadron nucleus scattering in the triple Pomeron region. These topics will be discussed in section 5.3.

4. Reggeon field theory at zero transverse dimensions

Several groups studied Reggeon field theory at transverse dimensions \( D = 0 \), or made appropriate approximations which effectively reduced the theory to zero transverse dimensions [15, 19, 36, 113, 119, 153]. Indeed, one finds that there are several reasons that make the calculation at \( D = 0 \) of particular interest: (1) Due to the small size of the slope, \( \alpha'_0 \), of the Pomeron pole, the shrinkage characterized by \( \alpha'_0 \ln s \) is only a small contribution to the diffraction slope up to very high energies. This is certainly true for nuclei (Schwimmer [153]) and to a lesser extent, although still considered as a possible first approximation, in the case of hadron scattering (Amati et al. [19], Gribov [113]); (2) At \( D = 0 \) the Pomeron propagator and the three point function can be evaluated exactly at \( E = 0 \) and the scaling behavior can be determined (Bronzan et al. [36]). The \( D = 0 \) calculations can be used as a testing ground for various approximation techniques used at \( D \neq 0 \); (3) In the lattice formulation of RFT the exact solution at \( D = 0 \) is actually the solution of the non-interacting theory [20, 36, 41, 119].

Gribov [113] had formulated a scheme whose first approximation takes into account the smallness of the Pomeron slope, \( (4\alpha' \sim 1/m^2) \), compared to the hadron radius, \( (R^2 \sim 1/4\mu^2) \). He then suggested that up to very high energies of the order of \( \ln s \sim R^2/4\alpha' \) the Pomeron Green’s function \( G^{1,1}(j, k^2) = [j - 1 + \alpha' k^2]^{-1} \) which in impact parameter and s space is given by:

\[
G^{1,1}(\log s, b) = \frac{\exp(-b^2/4\alpha' \log s)}{4\pi\alpha' \log s} \tag{4.1}
\]
will be approximated \((\alpha' \to 0)\) by
\[
G^{1,1}(\log \, s, \, b) = \delta(b). \tag{4.2}
\]
Hence, although not really working at \(D = 0\), the impact parameter dependence therefore becomes simple in this theory with an infinitely heavy \((\alpha' \to 0)\) Pomeron. The results are a factorized total cross section that rises like \((\ln \ln \, s)^2\), dominated by a pole and enhanced cuts. The ratio \(\sigma_{el}/\sigma_{tot} \to \text{const.}\), the average multiplicity, rises like \(\ln \, s (\ln \ln \, s)^2\) and \(\langle n(n - 1) \rangle \sim \langle (n)(\ln \ln \, s)^2 \rangle^2\) resulting in a multiplicity distribution wider than the Poisson distribution. The introduction of small \(\alpha'\) does not change much of these results at moderate energies. At higher energies and \(\alpha' \neq 0\) this theory has properties similar to the strong coupling scaling solution that was discussed in section 1.

The approximation \(\alpha' = 0\) which effectively reduces RFT to a one dimensional \((D = 0)\) theory, where only the “time” \(\tau\) is a variable, was studied by Schwimmer [153] in scattering on nuclei and by Amati et al. [19] in pp scattering (section 5).

The exact solution obtained at \(D = 0\) is identical to that of the single site dynamics in the lattice formulation of RFT (section 7); it is therefore of much interest and will be discussed here in some detail (Bronzan et al. [36], Jengo [119], Brower et al. [41]).

If we denote the boundary value of the Heisenberg field operators \(\psi(\tau)\) and \(\bar{\psi}(\tau)\) in eqs. (1.9) and (1.11) by \(\psi_{(0)} = a\) and \(\bar{\psi}_{(0)} = a^\dagger\), then the Hamiltonian at \(D = 0\) becomes
\[
H = H_0 + H_1 = \Delta_0 a^\dagger a + \frac{i}{2} \gamma a^\dagger (a^\dagger + a) a \tag{4.3}
\]
where \(a\) and \(a^\dagger\) satisfy the usual harmonic oscillator commutation relations and the bare vacuum satisfies \(a|0\rangle = 0\), and therefore \(H|0\rangle = 0\). Bronzan et al. [36] and Jengo [119] had shown that indeed the state with \(E = 0\) is the lowest energy state and all other eigenstates have real and positive energy. Equation (4.3) can be written in the form \(H = a^\dagger \bar{H} a\) where
\[
\bar{H} = \Delta_0 + i \frac{r_0}{\sqrt{2}} x \tag{4.4}
\]
and \(x = (1/\sqrt{2}) (a^\dagger + a)\).

The propagator at \(D = 0\) is given by
\[
G^{1,1}(E) = \int_{-\infty}^{\infty} dt \, e^{iE t} \langle 0|T(\psi(t)\bar{\psi}(0))|0\rangle = i\langle 0|a(E - H)^{-1} a^\dagger|0\rangle \tag{4.5}
\]

since \(aH = (N + 1)\bar{H} a\), where \(N = a^\dagger a\) one finds
\[
G^{1,1}(E) = -i \left\langle 0 \left| \left[ \bar{H} - \frac{E}{N + 1} \right]^{-1} \right| 0 \right\rangle. \tag{4.6}
\]
The eigenstates of \(\bar{H}\) are the eigenstates of the operator \(x\) (see for example ref. [41])
\[
|x\rangle = \pi^{-1/4} \exp(-x^2/2 - a^\dagger a/2 + x\sqrt{2}a^\dagger)|0\rangle \tag{4.7}
\]
and a complete set of these states is inserted in eq. (4.6) to obtain at \(E = 0\)
\[
G^{1,1}(0) = -i\langle 0|\bar{H}^{-1}|x\rangle(x|0\rangle = -i \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} \left( \Delta_0 + i \frac{r_0}{\sqrt{2}} x \right)^{-1} \tag{4.8}
\]
for \(\Delta_0 > 0\) one finds
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\[ G^{1,1}(0) = -i \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} \int_{0}^{\infty} dy \exp \left\{ -y \left( \Delta_0 + i \frac{r_0}{\sqrt{2}} x \right) \right\}. \]  

(4.9)

After the integration on \( x \) is carried out and the result analytically continued to \( \Delta_0 < 0 \) (it can be done as long as \( r_0 \neq 0 \)) the \( y \) integration can be approximated for \( \Delta_0 \rightarrow -\infty \) by the method of steepest descent:

\[ G^{1,1}(0) = -i \frac{2\sqrt{2\pi}}{|r_0|} \exp \left( \frac{2\Delta_0^2}{r_0^2} \right). \]  

(4.10)

For \( \Delta_0 \rightarrow -\infty \) only the energy of the first excited state approaches zero while all \( \epsilon_i \sim O(|\Delta_0|) \). In this limit and small \( E \), eq. (4.5) gives:

\[ G^{1,1}(E) = i \frac{\langle \varphi | a_i | \epsilon_i \rangle \langle \epsilon_i | a^\dagger | 0 \rangle}{E - \epsilon_i}. \]  

(4.11)

The eigenstate \( |\epsilon_i\rangle \) can be approximately constructed at \( \Delta_0 \rightarrow -\infty \) and the matrix elements, \( \langle 0 | a | \epsilon_i \rangle \) and \( \langle \epsilon_i | a^\dagger | 0 \rangle \), can be calculated. (\( |\epsilon_i\rangle \) has however infinite matrix elements for certain local operators.) Then, combining eqs. (4.11) and (4.10), the energy of the first excited state \( |\epsilon_i\rangle \) is found [36, 41]:

\[ \epsilon_i = \sqrt{\frac{2}{\pi}} \frac{\Delta_0}{|r_0|} \exp \left( -\frac{2\Delta_0^2}{r_0^2} \right). \]  

(4.12)

Similar result was obtained by using the equivalence of the problem with a Schrödinger equation [119]. The energy \( \epsilon_i \rightarrow 0 \), as \( \Delta_0 \rightarrow -\infty \). This sets a new small scale in the theory. In formulating RFT on the lattice, the inequality \( |\Delta_0| \gg \epsilon_i \), is used to approximate the single site dynamics by the two lowest states \( |0\rangle \) and \( |\epsilon_i\rangle \) (section 7).

As mentioned above there is more to be learned from the \( D = 0 \) calculation. Knowing the exact solution it can be then used as a laboratory for testing various approximation schemes (Bronzan, Shapiro and Sugar [36]). The exact solutions for any Green’s function at zero energy can be obtained. To study the renormalization group functions, critical exponents and the scaling behavior of the theory, it is enough to know the solution of \( G_{(0)}, G_{(0),}, \) and \( \partial G^{1,1}/\partial E \). Written as a function of \( h = r_0/\sqrt{2}\Delta_0 \) they are:

\[ iG^{1,1}(E = 0) = 2 \frac{\exp(-h^2)}{\Delta_0 h} \int_0^h \frac{dh'}{h'} \exp(-h'^2). \]  

(4.13a)

\[ G^{1,2}(E_{1,2} = 0) = -i \frac{2\sqrt{2} \exp(h^2)}{h^2} \int_0^h dh' \exp(-h'^2)[G^{1,1}(h)]^2. \]  

(4.13b)

\[ i \frac{\partial G^{1,1}}{\partial E} (E = 0) = -i \frac{\sqrt{2}}{h^2} \int_0^h dh' h' G^{1,2}(h'). \]  

(4.13c)

From these expressions the Green functions can be written in formal perturbation expansions which are asymptotic series in powers of \( h^2 \); for example,

\[ (\Delta_0 > 0) \quad iG^{1,1}(0) = \Delta_0^{-1} \sum_{n=0}^{\infty} (2n)!(n!)^{-1}(-h^2/4)^n \]  

(4.14a)
(Δ₀ < 0) \quad iG^{1,1}(0) = (8π)^{1/2}r₀^{-1} \exp(h^{-2}) + Δ₀^{-1} \sum_{n=0}^{∞} (2n)!(n!)^{-1}(-h^2/4)^n. \quad (4.14b)

Combining eqs. (4.13a,b,c) with eqs. (2.1a,b,d), normalized at \( E_N = 0 \), gives us the necessary wave function renormalization constants. The renormalization group functions defined in section 2 can now be written exactly from eqs. (4.13a,b,c) or in formal perturbation expansion using expressions like eqs. (4.14a,b). As seen in eq. (4.11), \( E = ε \) is the location of the renormalized Pomeron pole at \( D = 0 \) and the critical theory is obtained as \( Δ₀ → -∞ \) at fixed \( r₀ \). One finds at \( D = 0 \) that the exact \( β(g) \) has an infrared stable fixed point (at \( g = g₁ = 2 \)); this is certainly an encouraging result since when added to the fact that \( β \) has also an infrared stable point near \( D = 4 \), one is tempted to believe that such a zero also exists at the physical dimensions \( D = 2 \). The Pomeron is a simple pole at \( D = 0 \) and the exact values of the critical exponents are \( γ = 0, λ = 1, κ = 1 \) (the exponent \( κ \) will be discussed in section 7). The points \( g = g₁ = 2 \), as well as \( g = 0 \), are branch points of \( β(g) \). The asymptotic series of \( β(g) = \sum n c_n g^{2n+1} \) one gets from perturbation theory cannot be used to determine \( g₁ \). The perturbation theory can be used however to extract information on \( β(g) \) by using the Padé or Borel-Padé approximants. The critical indices as well as \( g₁ \) can be calculated with these techniques. However, at \( D = 0 \) the convergence is slow and in order to get good approximation, the power expansion has to be used up to order \( g^{11} \) (\( g^{13} \).

Learning about the merits of the Borel-Padé approximants at \( D = 0 \) Bronzan et al. [36] then used this technique at \( D = 2 \). It was found that the Borel-Padé approximants converge very well already at the three loop order. (The three loops - \( \mathcal{O}(g^7) \) calculations were done by Harrington [116].) The value for \( g₁ \) and the critical exponent are larger than those obtained from the power expansion.

The Borel-Padé approximants results and the critical exponents extracted [36] by this technique may be considered more reliable than the results from the power expansion. (The numerical values were cited in table 1 in section 1.) Note however that the claim that the Borel-Padé approximants converge much better at \( D = 2 \) than at \( D = 0 \) is based only on the agreement between the two and three loop calculations at \( D = 2 \).

5. Application of Reggeon field theory at the tree approximation level

The discussion of the attempts to estimate the value of the transition energy \( s_T \) in section 4.4 demonstrated the difficulties in obtaining a reliable estimate and at present we are left with a very wide range of values for \( s_T \). One still cannot say for sure whether multi-Pomeron exchange, and Pomeron interactions other than the first few terms in the perturbative approach will be important at energies say slightly above the ISR. One of the factors that might delay the onset of the scaling solution is the weakness of the triple Pomeron coupling, being manifested in the small ratio of the diffraction dissociation to the total cross section. Hadron nuclei scattering was explored as a possible experimental testing ground for RFT ideas in view of the fact that the strength of the effective Pomeron interaction is enhanced here, and thus hoped to be relevant to experiments at present energies. These enhanced contributions appear in tree graphs in which the triple Pomeron, \( r \), is being multiplied by a number proportional to the amount of nuclear matter in the nucleus \( gA^{1/3} \) [153]. The same study was extended also to hadron hadron scattering [19, 68, 142]. Here, however, \( r \) is enhanced only by the coupling of the Pomeron to a hadron, and these tree graph contributions may therefore be at most of a phenomenological importance in hadron hadron scattering only at energies higher than the relevant energies in the hadron nucleus scattering. The direct phenomenological aspects of these ideas are, however, subject to controversy, as will be discussed below.
5.1. Application of RFT to hadron nucleus scattering

Hadron hadron Regge phenomenology, together with a general knowledge of the nuclear structure, hopefully supply enough information to enable us to predict the features of hadron nucleus scattering at high energy. This in turn can be a useful tool for investigating such features as elastic rescattering and the above mentioned enhanced inelastic rescattering.

Several reviews on this subject are available, presenting the particle nucleus scattering in Regge theory, space-time picture of a “soft” field theory [144, 165], and the parton model [30]. Other approaches to this issue can be found in the literature as well [14, 103]. In this section we will discuss the application of Reggeon field theory to several problems in hadron nucleus scattering.

Simple pole contribution to particle nucleus total cross section (fig. 30a) is accompanied by multi-Pomeron exchange, described by non-planar [135] diagrams of the type of fig. 30b, which modify the customary Glauber’s picture [99, 101]. The resulting scattering amplitude in such a generalized Glauber’s elastic rescattering scheme resembles a scattering on a disc, whose radius is \( R = a_0 A^{1/3} \) (as long as \( a'Y \ll R^2 \) which is true for all accelerator energies), namely

\[
\sigma_T \sim 2\pi a_0^2 A^{2/3}
\]  

(5.1)

where \( a_0 = 1 \text{ fm} = 5 \text{ GeV}^{-1} \). In case of the inclusive cross section we have seen in section 3.2 that by applying the AGK rules, the only non-vanishing contribution is the single cut Pomeron exchange. Hence, if the average multiplicity produced by the cut Pomeron (fig. 30a) is proportional to \( \ln s \), then

\[
\langle n \rangle_A \sim A^{1/3} \ln s.
\]  

(5.2)

Only eq. (5.1) is a rather good qualitative description of the experimental data (Murthy et al. [145], Biel et al. [29]). \( \langle n \rangle_A \) however is definitely not proportional to \( A^{1/3} \). The average multiplicity has a milder dependence on \( A \) [46]. The contribution from particles produced at small angles \( \theta < 3.5^\circ \) is essentially independent of \( A \) (fig. 32a). Contributions from large angles introduce some \( A \) dependence (fig. 32b).

There exists an attractive possibility that energy momentum conservation constraints imposed on the AGK cutting rules will have a major effect on the result in eq. (5.2) [56]. In fact, it was argued that these constraints imposed on hadron nucleus scattering can account for the main features of the data.

Other possible modifications of Glauber’s picture were discussed by many authors [28, 112, 125, 130–131, 153, 171]. One of these ideas [153] will be discussed below. Although its phenomenological success is in doubt, at least at the level of its present implementation in RFT, it is being mentioned here due to its theoretical interest and its relevance to the analysis of the “classical” limit of RFT (which we will discuss in the latter sections of this report). It was suggested that at high energies, the picture should be modified rescattering not only of the incoming particle, as in fig. 30b, but also by rescattering of the produced particles. At high energies there are many such produced particles, and

![Fig. 30. (a) Pomeron pole contribution to particle nucleus total cross section. (b) Multi-Pomeron exchange contribution to particle nucleus total cross section. The \( n \) Pomeron-particle coupling blob denotes a non-planar coupling (see e.g. fig. 26).](image)
the square of the production amplitude (fig. 31) is given by the absorptive part of the particle–nucleus elastic scattering amplitude, which includes triple Pomeron interactions. This inelastic rescattering is in contrast to the original Glauber's picture in which the elastic rescattering resulted in multi Pomeron exchange with no Pomeron interactions (fig. 30b). Note that each triple Pomeron interaction of strength, \(r\), introduced into a given diagram, is accompanied by a factor \(A^{1/3}\). This factor will appear only in tree diagrams where the two Pomerons leaving the interaction vertex are attached to the nucleus and not to the projectile.

The connected part of the diagrams is given by the sum on \(N\) of all connected Green's functions \(G^{N,1}\) (fig. 33). This will be calculated with the \(r\bar{\psi}\psi^2\) term only in \(\mathcal{L}_1\) (eq. (1.11)) and without the interaction term \(r\bar{\psi}\psi\) which will not be accompanied by an \(A^{1/3}\) factor. The disconnected part will include the exchange of several of the structures shown in fig. 33 and this will amount to the exchange of disconnected \(G^{N,M}\) Green's functions calculated however with the interaction term \(r\bar{\psi}\psi^2\) only, resulting in inelastic and elastic rescattering.

The sum of all tree diagrams contributions to the elastic amplitude is easily obtained using a path integral formalism. It corresponds to the classical path given by the solutions of the classical equations of motion [12]. For example, the contribution of a connected diagram with \(N\) Pomeron attached to the nucleus is [153]:

\[
T_{1,N}(Y, B) = \int \frac{g^{N+1}}{N!} \prod_{i=1}^{N} \rho_A(y_i, b_i) G^{N,1}(Y, B, \{y_i, b_i\}) \prod_{i=1}^{N} dy_i d^2b_i. \tag{5.3}
\]

The nucleus is treated as a state of loosely bound nucleons, \(\rho_A(y, b)\) is the nuclear matter density normalized to \(A\), and \(g\) is the coupling of the Pomeron to a nucleon. If a path-integral representation is written for the Green's functions \(G^{N,1}\), and the summation over \(N\) is performed, then \((g^{N!/N!})\rho_A^N\) exponentiates. Adding now also the contributions of \(G^{N,M}(M > 1)\), which add disconnected contributions, since only the \(r\bar{\psi}\psi^2\) term will be employed in \(\mathcal{L}_1\), one finds for the sum of elastic and inelastic rescattering:

\[
1 - T^A(Y, B) = \sum_{M,N=1}^{\infty} T_{M,N} = g \int D[\psi] D[\bar{\psi}] \exp \left\{ \int [\mathcal{L}(\psi, \bar{\psi}) + g_A \rho(y, B-b)\psi(y, b) + g \rho(y, b)\bar{\psi}(y, b)] dy d^2b \right\}. \tag{5.4}
\]

In eq. (5.4) the coupling of \(n\) Pomerons to the particle line was approximated by \(N_n = g^{n!/n!}\) [146], the 1 on the left hand side was added for the missing (0, 0) term and \(\rho(B-b)\) is a profile function for the projectile. The classical path is given by the solution of the classical equations of motion derived from the Lagrangian including the \textquoteleft source\textquoteright terms \(gA\bar{\psi}\) and \(g_A\rho\psi\). For example, for \(\psi\) we have:

\[\text{Fig. 31. Contributions to particle nucleus total cross section resulting from rescattering of the produced particles. In Reggeon diagrams language they give rise to Pomeron interactions in the elastic particle–nucleus scattering amplitude.}\]
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Fig. 32. (a) The ratio between the average number of particles produced in \( \pi p \) and \( \pi \) nucleus scattering at 100 GeV/c versus \( \bar{v} = A\sigma_{p\pi}/\sigma_{\pi A} \), where \( \sigma_{p\pi} \) and \( \sigma_{\pi A} \) are the absorption cross section of the incident \( \pi \) on \( p \) and \( A \), respectively. \( \times \) \(-0^\circ < \theta_{\text{abs}} < 3.5^\circ \); \( \bullet \) \(-0^\circ < \theta_{\text{abs}} < 26^\circ \); \( \Delta \) \(-0^\circ < \theta_{\text{abs}} < 110^\circ \) [46]. (b) \( n_{A}/(n)_{\text{H}} \) summed on all angles versus \( \bar{v} = A\sigma_{p\pi}/\sigma_{\pi A} \) [46].

\[
\frac{\partial \psi}{\partial y} = \alpha' \nabla^2 \psi + \mu \psi - \frac{r}{2} \psi^2 - g \rho_{\pi}(y, b) \tag{5.5}
\]

where \( \mu = \alpha_0 - 1 \).

A simplified model for the densities is

\[
\rho_{\pi}(y, b) = \frac{A^{1/3}}{\pi a_0^3} \delta(y) \theta(R - |b|) \tag{5.6a}
\]

and for the projectile

\[
\rho(y, B - b) = \delta^2(b - B) \delta(Y - y). \tag{5.6b}
\]

If \( \alpha' \) is neglected at energies for which \( \alpha' Y \ll R \), the classical equations of motion can be solved exactly. The integration along the classical path in eq. (5.4) gives:

\[
T^\pi(y, B) = 1 - \exp\{-g_a U(Y, B)\} \tag{5.7}
\]

where \( U(Y, B) \) is the connected part given by:

\[
-g_a U(Y, B) = -g_a \frac{2\mu}{r} \left[ \frac{\theta(R - |B|)}{1 - G \exp(-\mu Y)} \right] \tag{5.8}
\]

where

\[
G = 1 - \frac{2\pi a_0^3 \mu}{rgA}. \tag{5.9}
\]

Fig. 33. The Reggeon's Green's function for one incoming, \( N \) outgoing Reggeons in the tree approximation.
The positive sign of $r$ assures the absorptive character of the multi-Pomeron exchanges. For small $r$ eq. (5.7) reduces to the simple eikonal scattering amplitude. For the particle–nucleus inelastic cross section eq. (5.7) gives:

$$\sigma_m^A = \pi a_0^2 A^{2/3} \left\{ 1 - \exp \left[ -\frac{2\mu s}{r(1 - G \exp(-\mu^2 Y))} \right] \right\}.$$  \hfill (5.10)

The inclusive distribution is obtained by cutting the diagrams of the type of fig. 34 in all possible ways and attaching the detected particle to the cut Pomeron (see fig. 34a,b). At very high energies the Abramovskii, Gribov and Kancheli [13] cutting rules (section 3.2) concluded that diagrams with multi-Pomeron exchange of the type of fig. 27b do not contribute to the inclusive cross section. When applied to the present problem, the AGK cutting rules imply that only the connected part (eq. (5.8)) contributes to the inclusive particle nucleus scattering cross section in the central region. Moreover, even from the connected part all graphs of the type of fig. 34b vanish and only fig. 34a survives [153].

Using eq. (5.8), convoluted with a pp amplitude (fig. 34a), one then finds that the average multiplicity is no longer proportional to $A^{1/3}$, and has an $A$ independent piece in addition to the contribution of the term in eq. (5.2). Though quantitatively the inelastic rescattering seems to point to the right direction, we find that in order to explain the data quantitatively, the strength of these contributions do not seem to be of the desired magnitude at present energies. In fact, estimates [142, 171] of the parameters in this model show that at present energies the exponential in eq. (5.10) is practically negligible compared to 1 and thus, the $A^{1/3}$ enhancement of the triple Pomeron coupling may not be sufficiently strong in order that Pomeron interactions be important up to very high energies. These estimates are, of course, subject to similar difficulties and uncertainties as those discussed in section 2.4.

It should be noted, however, that using only diffractive rescattering and the AGK rules over a limited subenergy at which the inelastic rescattering occurs is certainly an oversimplification of the matter. An improvement of the model can be considered by adding lower Regge trajectories in order to take into account some of the low subenergies effects [44], and introducing appropriate energy momentum conservation constraints into the calculation. This can then be tested also by confronting the data on the rapidity distribution of the produced particles in hadron nucleus scattering [34].

$n$-particle inclusive cross sections can be calculated using the above method, and long range correlations between the produced particles in the central region are found [153]. The data show a clear deviation from the Poisson distribution [46]. Using the cut Reggeon field theory formulated in section 3.4, the distribution of knocked-out nucleons was calculated [52]. Their average number is proportional to $A^{1/3}$ and the average number of mesons produced in association with a given number $N$ of knocked-out nucleons is independent of $A$ and $N$. Further measurements of these quantities will be of interest in providing more constraints and a check as to the relevance of RFT analysis to hadron nucleons scattering.

Somewhat less model dependent relations can be also derived [51]. A direct prediction of the
discontinuity rules used above is that the inclusive cross section in the central region is proportional to 
\( g_n U(Y) \). If the nucleus is black and therefore \( \sigma_{in}^{AA} \) is independent of the projectile \( a \), one finds that \( \langle n \rangle_{A} \) is proportional to \( g_n \) and therefore to \( \sigma_{nn}^{AA} \). The expression for \( T^A \) in eq. (5.7), independent of the details of \( U(Y) \), teaches us also about the relation between the ratio of the hadron hadron and hadron nucleus inclusive cross sections in the central region and the blackness of the nucleus. This form for the scattering amplitude \( T^A \) is obtained also in an iterative solution to the elastic unitarity relation [49], and it may have therefore a more general status than its above derivation. Using the fact that only the connected part contributes to the particle nucleus inclusive cross section in the central rapidity region one finds [51]:

\[
g_n U = \frac{\sigma_{in}^{AA}}{\pi R^2} \left[ \frac{1}{\sigma_{in}^{AA}} \frac{d\sigma^A}{dy} \right] \frac{1}{\sigma_{in}^{AA}} \frac{d\sigma}{dy} = \frac{\sigma_{in}^{AA} R_c}{\pi R^2} \tag{5.11}
\]

where \( \sigma_{in}^{AA} \) is the particle (a) nucleus (A) inelastic cross section and \( d\sigma^A/dy \) and \( d\sigma/dy \) are the inclusive cross sections of the nucleus, and of the nucleon, respectively. Equation (5.11) puts a strong constraint on the inelastic spectrum; it implies that for the nucleus, in order to become black, namely, \( T^A = 1 - \exp(-g_n U) \to 1 \), the ratio between densities, \( R_c \), has to be large.

If one does not assume for \( T^A \) the particular form of eq. (5.7), then only bounds on \( R_c \) can be derived. In case of multi-Pomeron exchanges with no Pomeron interactions [127]

\[
R_c \geq n + 1 + \frac{2^{n-1}(2r-1)}{1-r} \tag{5.12}
\]

where

\[
\frac{1}{2} \left( 1 - \frac{2^{2-n}}{2^{1-n}} \right) \leq r = \frac{\sigma_{DD}}{\sigma_T} \leq \frac{1}{2} \left( 1 - \frac{2^{1-n}}{2^{1-n}} \right). \tag{5.13}
\]

Again, we see that blackness is achieved only if \( R_c \) is large. On an even more general ground it can be argued that hadron-hadron and hadron-nucleus inelastic scattering must be different from each other [144]; a similarity will be very difficult to understand, since it implies the dominance of short range correlations and factorization, which is, however, incompatible with the blackness of large nuclei.

It is very unlikely that inelastic rescattering introduced through Reggeon field theory will provide a complete answer to the many important problems that arise in hadron nucleus scattering. Though the triple Pomeron coupling is enhanced, its enhancement is probably not strong enough to result in a dominant effect. A detailed quantitative estimate of the size and measurability of this effect needs further refinements; in particular, in order to take into account correctly low energy effects. If inelastic rescattering effects can indeed be measured in scattering on nuclei, a unique opportunity may be given for checking RFT ideas without the necessity to go to extremely high energies.

5.2. Tree graphs in hadron hadron scattering with \( \alpha(0) > 1 \)

The motivations for summing tree graph diagrams at intermediate energies in hadron nucleus scattering were discussed above. At these energies intercept renormalization due to Pomeron loops may be still insignificant. In hadron hadron scattering (fig. 35) the triple Pomeron coupling is enhanced in tree graphs by \( g \), the coupling of the Pomeron to the scattering particle; thus, the enhancement is milder than that in the nuclear case. Experimentally it has been found that indeed a typical \( r^2 g^2 \) Pomeron loop contribution is of the order of 20–40% of the typical \( rg^3 \) effect in the large mass diffractive production cross section [19, 142].
Demanding that the Pomeron intercept will retain a specific value $\alpha_0 = \alpha_{0c}$, results, as explained in section 1, in a critical theory with a renormalized Pomeron at $\alpha(0) = 1$. A perturbative approach to the ISR data requires that $\alpha_0 - 1 = 0.06 - 0.1 \ [53, 142]$ (depending on the details of the fit, the upper value may be a better choice). The theoretical value of $\alpha_{0c}$ is however subject to controversy. As discussed in section 2.5 Capella and Kaplan [53] found that $\alpha_{0c} - 1 = 0.01$, whereas a nonperturbative approach, which may be more adequate for this type of calculation gives $\alpha_{0c} - 1 = 0.04 - 0.07 \ [55]$. In any case, whether the bare Pomeron intercept $\alpha_0$ does equal $\alpha_{0c}$, and the asymptotic theory is critical, or if it satisfies $\alpha_0 > \alpha_{0c}$, and a certain disc mechanism (section 7) represents the asymptotic theory, one has to deal with a bare Pomeron with intercept above one. Inelastic and elastic rescattering in fig. 35 will certainly have a major effect on such an input and if indeed the asymptotic strong coupling solution is beyond experimental reach, then the study of tree graphs may have attractive phenomenological application.

An analysis of hadron scattering with a bare Pomeron intercept above one would necessarily have to deal with some kind of formalism which includes summations of an infinite number of diagrams, in order to restore $s$-channel unitarity, since each term separately unavoidably increases as a power of $s$.

The tree graphs contributions to proton proton scattering (fig. 35) were studied by Amati et al. [19]. The analysis was carried out for the case $\alpha' = 0$, observing that at the energies of interest, the shrinkage $\alpha' \ln s$ is not the dominant part in the diffraction slope. The Lagrangian is given in eqs. (1.9) and (1.11) (with $\alpha' = 0$):

$$\mathcal{L} = \frac{1}{2} \frac{\partial}{\partial y} q - \mu pq - \frac{r}{2} p(p + q)q \tag{5.14}$$

where $q = i \psi, p = i \bar{\psi}, \mu = -\Delta = \alpha(0) - 1, y = ir$. Two source terms, which are the analog of the sources in eq. (5.4), should be added to the Lagrangian in eq. (5.14). These are $g_1 p$ and $g_2 q$, where:

$$g_1 = \beta_1(b) \delta(y), \quad g_2 = \beta_2(b - B) \delta(y - Y). \tag{5.15}$$

The partition function (normalized to $Z(0, 0) = 1$),

$$Z(g_1, g_2) = N \int dp \ dq \ \exp \left\{ i \int \left[ \mathcal{L}(p, q) + g_1 p + g_2 q \right] \ d\tau \ dB \right\} \tag{5.16}$$

is the elastic $S$ matrix, if we assume that the Pomeron interacts with the source in an uncorrelated way. The sum of all tree diagrams is given, as discussed in section 5.1, by the classical path in eq. (5.16)

$$S(Y, B) = \frac{\exp\{A_c(g_1, g_2)\}}{\exp\{A_c(0, 0)\}} \tag{5.17}$$
The extremum of the action is found by integrating along the classical path \( \{ p(y, b), q(y, b) \} \), where \( q_{cl} \) and \( p_{cl} \) are the solution of the classical equations of motion

\[
\dot{q} = \mu q + r(pq + \frac{1}{2} q^2) - \beta_2 \delta(y) \tag{5.18a}
\]

\[
\dot{p} = -\mu p - r(\frac{1}{2} p^2 + pq) + \beta_2 \delta(Y - y). \tag{5.18b}
\]

The classical action is given by:

\[
A_{cl} = \int dy \, d^2 b \mathcal{L}(q_{cl}, p_{cl}). \tag{5.19}
\]

The path integral formulation and the classical solution to RFT will be applied again later in studies of the inclusive cross section and defining the theory on the lattice. It will be therefore useful to follow in detail the solution to the present problem. Our approach will be however slightly different from that of ref. [19], in order to be able to establish a common scheme with that discussed later in section 5.3.

Since the \( b \) dependence in eq. (5.18) enters only through \( \beta_2 \) and \( \beta_1 \), it can be factorized out and introduced through appropriate boundary conditions:

\[
p(y, b) = P(y, b) \theta(Y - y)
\]

\[
q(y, b) = Q(y, b) \theta(y) \tag{5.20a}
\]

where

\[
P(Y, b) = -\beta_2 (B - b)
\]

\[
Q(0, b) = -\beta_1 (b). \tag{5.20b}
\]

These boundary conditions remove the \( \delta \) functions in eq. (5.18) from the equations of motion of \( P(y) \) and \( Q(y) \). The Hamiltonian derived from eq. (5.14) including the sources is:

\[
H = \mu pq + \frac{1}{2} r pq(p + q) - \beta_1 \delta(y)p + \beta_2 \delta(Y - y)q. \tag{5.21}
\]

Thus, the action for an orbit of given energy is given by:

\[
A_{cl} = \int d^2 b \mathcal{A}_{cl} = \int d^2 b \left[ \int_0^Y dy (p \dot{q} - EY + \beta_1 p(0) + \beta_2 q(Y)) \right] \tag{5.22}
\]

where a total derivative was added to \( \mathcal{L} \), and

\[
E = \mu PQ + \frac{1}{2} r PQ (P + Q). \tag{5.23}
\]

Using the boundary condition in eq. (5.20b) we finally have

\[
\mathcal{A}_{cl} = \int P \, dQ - EY - P(Y) Q(Y) \tag{5.24}
\]

where the first term is the area under the orbit of energy \( E \) in the \( PQ \) plane. A typical orbit and the structure of the \( PQ \) phase is shown in fig. 36. The lines

\[
P = 0, \quad Q = 0, \quad P = -Q + 2 \mu / r \tag{5.25}
\]
are zero energy lines. The fixed points where the “velocity” \( \dot{Q} \) and the “force” \( \dot{P} \) vanish are:

\[
\{Q, P\}_{F.P.} = (0, 0), \left( -\frac{2\mu}{r}, 0 \right), \left( 0, -\frac{2\mu}{r} \right), \left( -\frac{2\mu}{3r}, -\frac{2\mu}{3r} \right).
\]  

A typical orbit of positive energy is shown in fig. 36 for the case \( \beta_1, \beta_2 > 2 \mu/r \). The total “time” \( Y \) spent in the motion can be easily found by using (5.23) to find \( P(E, Q) \), which is then inserted in (5.18a), and finally

\[
Y = \int_{-\beta_1}^{\beta_1} \frac{dQ}{Q} = \int_{-\beta_1}^{\beta_1} \left[ \left( \mu Q + \frac{r}{2} Q^2 \right)^2 + 2rQE \right]^{-1/2} dQ.
\]  

Since during this motion \( Q(y) < 0 \) and \( Q(y) \sim E \) as \( E \to 0 \), we see in eq. (5.27) that in this limit

\[
Y = (I_1 + I_2) \log E.
\]  

The two equal contributions \( I_1 = I_2 = -1/\mu \) come from the integration region near \( Q = -2\mu/r \) and the end point \( Q(y) \sim E \to 0 \). These are the regions near the fixed points \( (P, Q) = (0, -2\mu/r), (-2\mu/r, 0) \) in fig. 36. If one demands a long time of travel, \( Y \), then a trajectory of low energy, \( E \), is required. At the fixed points the “velocity” \( \dot{Q} \) and the “force” \( \dot{P} \) vanish and they are small in the nearby region. For this reason, a trajectory of low \( E \) will spend most of the traveling time \( Y \) near the two fixed points, in an equal amount at each of them.

The action \( \tilde{A}_{cl} \) in eq. (5.24) at large \( Y \) receives its main contribution from the area under the orbit, \( P(Q) \), since \( EY \sim E \log E \to 0 \), and \( P(Y)Q(Y) \sim E \to 0 \). This area, however, is a constant and for \( \beta_1, \beta_2 > 2 \mu/r \) we have asymptotically

\[
\tilde{A}_{cl} = \frac{1}{2} (2\mu/r)^2.
\]  

The integration in eq. (5.22) on \( b \) extends to the regions of large \( |b| \), and large \( |b - B| \). In those regions \( \beta_1 \) and \( \beta_2 \) are small \( (\beta_1, \beta_2 < 2\mu/r) \) and trajectories different from the one in fig. 36 determine \( \tilde{A}_{cl} \) in eq. (5.24). (See next section for a discussion about other types of orbits which appear here and in the triple Pomeron region [142].) \( \tilde{A}_{cl} \) remains, however, \( Y \)-independent in the asymptotic limit \( Y \to \infty \), so that after integrating eq. (5.19) we have a constant \( S(Y, B) \) in eq. (5.17) approaching from below and resulting in an asymptotic grey disc.
Thus, starting with an amplitude growing like $s^n (\alpha(0) > 1)$ one finally finds that tree graphs alone, representing elastic and inelastic rescattering, prevent a power growth. The Pomeron with intercept above one turns into a grey disc due to Pomeron interactions even before introducing any loop corrections. Two questions are of interest now: (a) what are the quantum corrections to the classical path? namely, what is the effect of introducing loops? (b) what happens in the case $\alpha' \neq 0$, where the problem is no longer effectively one dimensional ($D = 0$)? Both problems were answered in a series of papers by Amati et al. [15, 20, 21] (section 7).

5.3. Inclusive cross sections in the tree graph approximation

The applications of RFT in the tree graph approximation to inclusive cross sections will be discussed in this section. The bare Pomeron intercept is set above one ($\mu > 0$) for the same reasons explained in the previous sections (5.1, 5.2), and the energies discussed here are in the region where $\alpha'$ can be neglected, namely, $\alpha' \ln s \ll R^2$, where $R$ is a typical radius of the scattering particles.

5.3.1. The triple Regge region

In the strong coupling solution, the inclusive cross section in the triple Regge region (sections 2.3 and 3.1) has a contribution to the inclusive sum rule, which is less than the total cross section [7, 59, 93]. Thus, the decoupling theorems and the other inconsistencies of models with weak Regge cuts [37, 122] were avoided when $\alpha_0 = \alpha_{oc}$. The consistency of the theory with $s$-channel unitarity for $\alpha_0 > \alpha_{oc}$ has been studied recently and will be discussed in detail in section 7. For the total cross section a disc mechanism seems to be able to avoid conflicts with the Froissart bound. It is desirable to understand now what happens with other $s$-channel constraints in the case of $\alpha_0 > \alpha_{oc}$. In particular, it is interesting to find out whether one is faced again with problems in the triple Regge region. The issue was studied (Moshe and Paige [142]) at the classical level only. At this level, we have seen in section 5.2 that starting with a Pomeron intercept $\alpha_0 = 1 + \mu > 1$, the sum of all tree graphs forces $S(Y, B)$ to increase up to a finite limit (Amati et al. [19]), independent of any particular value chosen for $\mu > 0$.

Consider the inclusive reaction $a + b \rightarrow c + \text{anything.}$ The independent variables are (fig. 37):

$$s = (p_a + p_b)^2 = m^2 e^y$$

$$M^2 = (p_a + p_b - p_c)^2 = m^2 e^n$$

$$t = (p_a - p_c)^2$$

(5.30)

where

$$m^2 = O(1 \text{ GeV}^2).$$

Fig. 37. Tree graphs in the triple Regge region. The time $\tau = -i\eta = -i \log M^2$ is held fixed and the “energy” ($E = 1 - j$) is not conserved.
The inclusive cross section in the triple Regge limit is given by a particular $M^2$ discontinuity of the $a + b + \bar{c} \rightarrow a + b + \bar{c}$ maximum helicity flip amplitude in the kinematic region $\eta \gg 1$, $Y - \eta \gg 1$. RFT rules for the triple Regge region were derived in refs. [7 and 58] (section 3.1).

It will be shown in this section that the cut-Pomeron formalism (section 3.4) provides a direct and transparent way of formulating the rules for the Reggeon calculus in the triple-Regge region. These rules can be expressed as a functional integral representation for the generating functional. In particular, they make it again apparent why energy is not conserved at the branching vertex (section 3.1). The formulation of the problem, employing the functional integral, provides for the analysis of the triple Regge region a scheme that can be studied at the classical as well as at the quantum level. The analysis at the classical level (to be discussed below) is interesting because of two reasons: First, its numerical, non-asymptotic solution may have a phenomenological importance at present energies; second, the classical solution may give some very useful insight into the problem when the quantum level is studied.

A typical Reggeon field theory graph contributing to $d\sigma/dy dt$ is shown in fig. 37. Pomeron propagators and interactions in this graph can be attributed to $\psi_+$, $\psi_-$ and $\psi_c$ Pomeron fields. Every such graph must contain a cut Pomeron which at $y = \eta$ couples to a plus and a minus Pomeron, since if no cut Pomeron is extended up to $y = \eta$, the missing mass would be less than $e^\eta$. This vertex at $y = \eta$ is called in [7] the branching vertex. For $y < \eta$ all of the interactions contained in eq. (3.17) can occur, but for $y > \eta$ only the interactions

\[ \mathcal{L}_1 = i\frac{r}{2} \bar{\psi}_c(\bar{\psi}_+ + \psi_+)(\bar{\psi}_- + \psi_-) - i\frac{r}{2} \bar{\psi}_-(\bar{\psi}_+ + \psi_+)(\eta, b) \psi_-(\eta, b) \psi_-(\eta, b) \exp \left\{ \int_0^Y dy \int d^2b \mathcal{L}_{\text{eff}} \right\} \tag{5.32} \]

where $r = iG$, are permitted. Any of the other interactions would involve a cut Pomeron extending beyond $y = \eta$, and hence, a missing mass greater than $e^\eta$. Thus, following steps similar to those in section 5.2 the cut Reggeon field theory rules for the triple-Regge region are summarized by the generating functional

\[ Z(Y, \eta, b, b_1, b_2, b_3) = \int \mathcal{D}[\psi_c] \mathcal{D}[\psi_i] \mathcal{D}[\psi_-(\eta, b)] \bar{\psi}_+(\eta, b) \psi_-(\eta, b) \exp \left\{ \int_0^Y dy \int d^2b \mathcal{L}_{\text{eff}} \right\} \tag{5.33} \]

Here $\mathcal{L}_0$ is given in eq. (3.16) and the $\eta$-dependent interaction Lagrangian is given by:

\[ \mathcal{L}_i(\eta) = i \frac{r}{2} \bar{\psi}_c(\bar{\psi}_+ + \psi_+)(\bar{\psi}_- + \psi_-) + \theta(\eta - y) \left\{ \frac{r}{\sqrt{2}} \bar{\psi}_c(\bar{\psi}_+ + \psi_+) \psi_+ \right. \]

\[ + \frac{r}{\sqrt{2}} (\bar{\psi}_c \bar{\psi}_+ \psi_- + \psi_c \bar{\psi}_+ \psi_-) + ir \bar{\psi}_c(\bar{\psi}_+ + \psi_+) - ir \bar{\psi}_c \psi_-(\bar{\psi}_+ + \psi_-) \right\}. \tag{5.34} \]

The $\psi_c \bar{\psi}_+ \bar{\psi}_-$ factor in eq. (5.32) forces each graph to have a cut-plus-minus vertex at $y = \eta$. It also insures the correct connectedness structure, since the only source for Pomerons is at $b$, and the only sinks for plus and minus Pomerons are at $b_1$ and $b_2$, respectively (fig. 37). Because of the $\theta(\eta - y)$ in $\mathcal{L}_i(\eta)$, the Hamiltonian is time-dependent at $y = \eta$, and hence the "energy" $E = 1 - j$ is not conserved [7, 58, 142] at that point as explained in section 3.1. In deriving eq. (5.32) it was assumed, following ref. [19], that the Gribov vertices $N^i_\alpha$ for coupling $n$ Pomerons to the external particles can be approximated by an eikonal form [146].
The Fourier transform of eq. (5.32) gives the sum of all cut Reggeon field theory graphs with external transverse momenta \( k_1, k_2 \) and \( k_3 \). Since \( Z \) is translation invariant, we have:

\[
(2\pi)^2 \delta^2(k_1 + k_2 + k_3) \tilde{Z}(Y, \eta, k_1^2, k_2^2, k_3^2) = \int d^2b_1 d^2b_2 d^2b_3 \exp\{i(k_1 \cdot b_1 + k_2 \cdot b_2 + k_3 \cdot b_3)\}Z(Y, \eta, b_1, b_2, b_3).
\]

Finally, the inclusive cross section (fig. 37) is clearly proportional to \( \tilde{Z}(Y, \eta, t, t, 0) \) and the correct normalization is [96, 142]:

\[
\frac{d\sigma}{d\eta \, dt} = \frac{r}{16\pi} \tilde{Z}(Y, \eta, t, t, 0).
\]

Equations (5.32)–(5.36) contain the full complexity of the Reggeon quantum field theory. Following ref. [19] we make two approximations possibly appropriate at moderate values of \( Y \): (1) since the triple-Pomeron coupling is much smaller than the Pomeron coupling to protons, as discussed in section 5.2, only the Pomeron tree graphs are kept. This sometimes is referred to as the classical approximation since it reduces the problem to that of solving a classical field theory. Of course it can be valid only if \( Y \) is not too large since asymptotically the fully enhanced graphs dominate. (2) only the case \( \alpha' = 0 \) is considered, which may be a reasonable approximation for present accelerator energies. While this has certainly a drastic effect on the quantum theory, its effect on the classical theory discussed here is less severe. In this approximation eq. (5.32) reduces to

\[
Z_{cl} = \psi_{+cl}(\eta, b) \tilde{\psi}_{+cl}(\eta, b) \tilde{\psi}_{-cl}(\eta, b) e^{4\alpha'}. \tag{5.37}
\]

where \( \psi_{cl} \) are the classical solutions of the fields in the Lagrangian in eq. (5.34) at time \( y = \eta \), and the classical action is:

\[
A_{cl} = \int^y_0 dy \int d^2b L_{cl}. \tag{5.38}
\]

The approximations made above are more adequate for the hadron nucleus scattering case [153]. In addition to the approximations above, at large atomic number \( A \) all interactions of the type \( \psi \tilde{\psi}^2 \) are neglected for the same reasons discussed in section 5.1 (fig. 38). The classical equations of motion can be solved exactly for \( \psi_+(y, b) \), \( \psi_-(y, b) \), and \( \psi_0(y, b) \) [142], and for the inclusive hadron nucleus scattering cross section \( (h + A \rightarrow h + \text{anything}) \) in the triple Regge region one obtains:

\[
\frac{d\sigma}{d\eta \, dt} = \frac{A^{2\mu} a_0^2}{4\sqrt{2}} \bar{\beta}^2(t) \exp\{-2\mu(Y - \eta)\} \frac{(1 - c e^{-\mu\eta})^3}{(1 - c e^{-\mu\eta})^4} \exp\left\{-\frac{4\mu\beta}{r} (1 - c e^{-\mu\eta} - 1)\right\}. \tag{5.39}
\]

Fig. 38. A typical tree graph contribution to the hadron nucleus inclusive cross section in the triple Pomeron region. Since all vertices are of the type \( n\tilde{n}\tilde{\phi}^2 \), the triple Pomeron coupling is effectively enhanced. \( r \rightarrow r A^{1/3} \).
where $c = 1 - 2\mu/r\beta_x$. The same notation as in section 5.2 was used for the parameters in eq. (5.39). $\bar{\beta}(t)$ is the Fourier transform of sharply peaked $\beta(b)$, and $\beta \equiv \beta(b = 0)$. In the limit $r \to 0$ ($c \to -2\mu/r\beta_x$), eq. (5.39) reduces to the usual triple Pomeron pole graph accompanied by elastic rescattering only.

We see from eq. (5.39) that at large $A$ the inclusive cross section in the triple Regge region is proportional to $A^{3/3}$ with calculable corrections. Future measurement of the $A$ dependence of $d\sigma/dt d\eta^2$ may be an interesting experimental test of the relevance of the classical approximation to present laboratory energies. There is, however, an experimental difficulty in measuring $p + A \to p + X$ at $x_F \to 1$ and low $t$ since it is quite impossible to separate the scattered proton from the beam. Another interesting calculation would be to add a lower trajectory Reggeon in the above scheme and predict the results of the new experiments of $\Lambda^0$ production on nuclei [117]. In these experiments the neutral produced particle can be easily separated by bending the beam.

Applying the above technique to hadron hadron scattering is somewhat more complicated. There is no reason, of course, to neglect the $r/\alpha_t r^2$ term in the interaction Lagrangian, and the whole set of equations of motion based on $\mathcal{L}_{\text{eff}}$ in eqs. (5.33) and (5.34) has to be solved. We will stay, however, within the approximation $\alpha' = 0$ which is still reasonable at not too high energies [19], admittedly, it is a less good approximation than setting $\alpha' = 0$ in the nuclear case.

A partial diagonalization of the equations of motion reduces the complexity of the problem significantly [142]. Using notations similar to those in section 5.2

$$q_\pm = \pm i\psi_\pm, \quad p_\pm = \pm i\bar{\psi}_\pm, \quad q_c = \psi_c, \quad p_c = \bar{\psi}_c$$

results in the elimination of all factors of $i$ in the classical equations, and it emphasizes the analogy with classical mechanics. Furthermore, defining at $y < \eta$

$$q_+(y, b) = q_-(y, b) = -\frac{1}{\sqrt{2}} q_c(y, b) = Q(y, b)$$

and

$$p_+ (y, b) + p_- (y, b) + \sqrt{2}p_c(y, b) = P(y, b)$$

reduces the problem to a single set of equations of motion for $P(y, b)$ and $Q(y, b)$ with an energy $E_3$

$$E_3 = \mu PQ + \frac{1}{2} r PQ(P + Q).$$

For $y > \eta$ the $\psi_\pm$ and $\bar{\psi}_\pm$ fields are decoupled and satisfy uncoupled equations of motion. The energies of the decoupled systems are:

$$E_1 = \mu p_+ q_+ + \frac{1}{2} r p_+ q_+(p_+ q_+)$$

$$E_2 = \mu p_- q_- + \frac{1}{2} r p_- q_-(p_- + q_-).$$

The new interesting ingredient here is the fact that energies are not conserved at $y = \eta$, as discussed in section 3.1 and above, namely,

$$H(y < \eta) = E_3 \neq E_1 + E_2 = H(y > \eta).$$

The $P, Q$ phase space contains therefore a discontinuous "jump" at time $y = \eta$ as shown in fig. 39 (for the case $\beta_1 = \beta_2$). The classical orbits in this plane, determined by $\mathcal{L}_{\text{eff}}$ (eq. (5.33)), depend on the relation between $\beta_1 = \beta(b - b_2)$, $\beta_{1,2} = \beta(b - b_{1,2})$, and the parameter $2\mu/r$. (Two classical paths are shown in fig. 39 in the simple case $\beta_1 = \beta_2$.) Each possible classical orbit in the path integral (eq. (5.32))
Fig. 39. The $P, Q$ phase space. The classical motion continues with energies $H(y < \eta) = E$, and $H(y > \eta) = E + E_i (= E_{ij})$. At $y = \eta$ the energy is not conserved. $H(y < \eta) \neq H(y > \eta)$ at a given point $(P, Q)$. The energy vanishes along the dotted lines at times $y < \eta$ ($E_i = 0$) and $y > \eta$ ($E_{ij} = 0$). Two classical orbits are shown: (a) is near the fixed point $(0, 4\mu/r)$ at time $y = \eta$. (b) is near the fixed point $(2\mu/r, 0)$ at time $y = \eta$. Orbit (a) has a smaller $A_{0,0}$ determines a contribution to $Z(\eta)$, and finally, $d\sigma/d\eta dt$ is found from eq. (5.36). In general $\beta_1 \neq \beta_2$ and there are three distinct contributions determined by four different paths [142]. The asymptotic behavior of each of these contributions shows a decrease like a power of $S/M^2$ or $M^2$. Thus, starting with a bare Pomeron with intercept above one ($\mu > 0$), the absorptive effect of the Pomeron exchanges and interactions in $L_1(\eta)$, introduced through tree graphs only, resulted in the elimination of the power increase of the bare triple pole graph.

It is easy to understand qualitatively the mechanism which reduces the power increase of the cross section into a power decrease, without following the detailed calculation. The free field ($r = 0$) single pole ($\beta$ small) amplitude is found in eq. (5.37) from

$$\psi(y) \sim \beta e^{\mu y}, \quad \bar{\psi} \sim \beta e^{\mu(y-x)}.$$  

Hence, the motion in the $pq$ phase space in this case is along an hyperbola and $d\sigma/d\eta dt \sim \psi(\eta)\bar{\psi}(\eta) \sim q(\eta)p^2(\eta)$ grows like $e^{2\mu y}$ at fixed $\eta$. This happens since the hyperbola's path moves away from the origin like $e^{\mu y}$ as $Y \to \infty$. The action is zero in the absence of multi-Pomeron exchanges and there are no fixed points other than the origin in the case $r = 0$. The situation changes drastically as multi-Pomeron exchanges and Pomeron interactions are added. The classical orbit is then forced to pass near the fixed points at the time $y = \eta$ (see fig. 39); thus, $p(\eta)$ is of order $E \to 0$ and $q \to 2\mu/r$, or $q(\eta)$ is of order $E \to 0$ and $p \to 2\mu/r$. In each case $d\sigma/d\eta dt \sim qp^2$ approaches zero due to the smallness.
of $E$, which decreases typically as an exponential of $\eta$ or $(Y-\eta)$, and therefore the cross section decreases as a power of $M^2$ or $S/M^2$. The action in eq. (5.37) gives a constant at large $Y$ and does not affect the above calculation.

Since $S(Y,B)$ at the tree graph level (section 5.2) approaches a constant, the inclusive sum rule is satisfied in the tree graph approximation. Of course, the true asymptotic behavior of RFT is determined by graphs involving Pomeron loops. For the total cross section, inclusion of loop graphs and $\alpha' \neq 0$ produces only a logarithmic modification of the energy dependence of the sum of tree graphs [20, 21]. If the same is true for the triple Regge region, then the inclusive sum rule will also be satisfied at the quantum level.

As to the relevance of the asymptotic solution to present data, one finds that the experimental data imply [142] that asymptopia is at $Y \geq 50$. Thus, even if rapidities $Y \sim 50$ were of interest, Pomeron loop graphs would then presumably be important, and the quantum effects should be taken into account for either $\alpha_0 = \alpha_{\infty}$ or $\alpha_0 > \alpha_{\infty}$ (section 7). This does not mean that the classical approximation in (5.37) is irrelevant for moderate energies. It simply says that the asymptotic solution is not a good approximation for realistic values of the parameters, and that other, presumably numerical methods, should be used in the calculation of $Z(\eta)$ in (5.37). The formulation of the problem in this section provides sufficient tools for such an interesting numerical analysis in the future. A different quantitative estimate of the effect of Pomeron interactions in hadron nucleus scattering at moderate energies is given in [120].

The perturbative approach to RFT discussed in section 2.5 offered an attractive scheme for the study of the inclusive cross section in the triple Regge region [55]. The bare triple Pomeron coupling $r$ in this scheme is not independent of $k^2$ as in eq. (5.34) above. Moreover, the strength of the absorptive corrections is very sensitive to the detailed $k^2$ dependence of $r(k_1,k_2,k_3)$. The recent measurement [89] of the strength of the inclusive cross section $pp \rightarrow px$ in the triple Regge region (see table 2 in section 2) raises however some difficulties with this approach [55].

### 5.3.2. The central rapidity region

As discussed in the previous sections, a theory with a bare Pomeron intercept above one ($\mu > 0$) has to provide a scheme for summing an infinite number of graphs contributing to the scattering amplitudes. The summation of tree graphs in sections 5.2 and 5.3.1 had shown that simple problems with $s$-channel unitarity are avoided at this level due to Pomeron interactions. Indeed, in the central rapidity region Pomeron interactions are necessary. If $r = 0$ and $\mu > 0$, the AGK rules (section 3.2) will imply the dominance of the single Pomeron exchange in $d\sigma/dy$, and therefore, an increase like $s^\mu$ of this cross section as $s \rightarrow \infty$, hence, a violation of the inclusive sum rule. Introducing Pomeron interactions treated at the tree graph level is sufficient [68] in order to force $d\sigma/dy \rightarrow \text{const}$. Other aspects of the central rapidity region and the results for the $k$ particles inclusive distributions will also be presented in this section. The discussion in sections 5.1, 5.2 and 5.3.1 provides the necessary tools for this section and we can almost directly read its results.

The impact parameter representation of the imaginary part of the elastic amplitude in fig. 17, assuming an eikonal form for the Pomeron coupling to the particle line [68], is:

$$ -T(Y,B) = S(Y,B) - 1 = \sum_{n,m} \frac{S^n}{n!} G_{(Y,B)}^{nm} \frac{f^m}{m!} $$

(5.45)

In order to calculate the inclusive cross sections the cut Reggeon field theory Lagrangian in eqs. (3.16) and (3.17) is employed. The new ingredient here, the coupling of the cut Pomeron to the detected
particle in a Mueller type diagram, is introduced by adding a mass insertion, $\xi(b, y)\bar{\psi}_c\psi_c$, in the Lagrangian. Using the identity enforced by s-channel unitarity between the cut and uncut Green's function (section 3.4) we have:

$$
T_c(\xi) = \frac{1}{2} \sum_{n, m, l, r} g^{n_r+n_l-(\sqrt{2}g)^{n_l}} G^{n_l, n_r}_\xi f^{m_r+m_l-(\sqrt{2}f)^{m_l}} m_r! m_l! m_c! \\
= 1 - S + \frac{1}{2}(\exp(-A_c(\xi, g, f, Y, B)) - 1)
$$

where $S$ is given in eq. (5.45) and $-A_c(\xi)$ is the action calculated from $\mathcal{L}$ in eqs. (3.16) and (3.17), including the mass insertion $\xi\bar{\psi}_c\psi_c$.

The $k$ particles inclusive cross section is now found by taking the $k$ functional derivative with respect to $\xi(b, y)$ [68], e.g.,

$$
\frac{1}{\rho} \frac{d\sigma}{dv} (v, V) = \left[ \frac{\delta T_c(\xi)}{\delta \xi(v)} \right]_{\xi=0} = -\frac{1}{2} \exp(-A_c(\xi)) \left[ \frac{\delta A_c(\xi)}{\delta \xi(v)} \right]_{\xi=0}
$$

$$
\frac{1}{\rho^2} \frac{d^2\sigma}{dv_1 dv_2} = \left[ \frac{\delta^2 T_c(\xi)}{\delta \xi(v_1) \delta \xi(v_2)} \right]_{\xi=0} = \frac{1}{2} \exp(-A_c(\xi)) \left[ \frac{\delta A_c}{\delta \xi(v_1)} \frac{\delta A_c}{\delta \xi(v_2)} - \frac{\delta^2 A_c}{\delta \xi(v_1) \delta \xi(v_2)} \right]_{\xi=0}
$$

where $\rho$ is the bare particle density. Since $S$ in eq. (5.46) is independent of $\xi$, the only quantity we have to calculate is $A_c(\xi)$. At $\xi = 0$ unitarity implies $T_c(\xi = 0) = 1 - S$, thus $A_c(\xi = 0) = 0$. We also have

$$
\delta A_c(\xi) / \delta \xi(v) = -\bar{\psi}_c(v, \xi) \psi_c(v, \xi)
$$

and higher derivatives of $T_c(\xi)$ will be obtained from higher order derivative of the fields with respect to $\xi$. The main task in finding the inclusive cross sections at the tree graph approximation is therefore to solve the classical equations of motion for the fields and determine their $\xi$ dependence. The problem here is simpler than in the previous section. Here one can use at any $y$ the equality (5.41a), reducing the problem to finding $Q(y, b)$. Moreover, in section 5.3.1 we had different sources, $\beta(b - b_1)$, and $\beta(b - b_2)$, and no source in the equations of motion of $P_\pi, P_\pi$, and $P_\pi$, respectively (see $\mathcal{L}_{\text{eff}}$ in eq. (5.33)). In the present problem $\beta_\pi = \beta_\pi = \beta_\pi \sqrt{2}$, thus we can have, in addition to (5.41a), $P_\pi = P_\pi = -P_\pi \sqrt{2} = P(y, b)$ instead of eq. (5.41b). The symmetry in the problem also implies that $P(y, b) = Q(Y - y, B - b)$.

A typical contribution to the single particle inclusive cross section in the central region is shown in fig. 40. When compared to fig. 34a it is clear that the renormalization of the particle Pomeron vertex here is identical to that we had in section 5.1. Thus, setting $\alpha' = 0$, or alternatively, the transverse dimensions $D = 0$, we can use the results in section 5.1 and find for $Q(y, b)$ an expression similar to $U(b, y)$ in eq. (5.8). The inclusive cross section is given by a convolution of two such expressions (fig. 40):

$$
d\sigma / dy ~ Q(y) Q(Y - y).
$$

It approaches a constant at $yu \gg 1$ and $\mu(Y - y) \gg 1$, hence, there are no conflicts with the inclusive sum rule at this level.

Even with $\alpha' = 0$ the two particles inclusive cross section in the central region reveals a complex structure [68]. The AGK cutting rules, incorporated here in the interaction Lagrangian (3.17), and in the sources, allow four different types of contributions (figs. 41a–d) at the tree graph level. Note that the diagram in fig. 41a contains in between the detected particles lines, a bare Pomeron singularity. This implies that its asymptotic behavior after the $y_1$ and $y_2$ integrations are carried out, is $s^{\alpha} \mu > 0$, where $s$ is
therefore fig. 41a by itself violates the second inclusive sum rule. However, when all four contributions are summed up, delicate cancellations between fig. 41d and fig. 41a, forced by the AGK rules, remove this $s^\alpha$ behavior [68], and the consistency with the sum rules is restored at the tree graph level.

The above results were obtained at $D = 0 (\alpha' = 0)$; the picture with $\alpha' \neq 0$ and $D = 2$ is somewhat more complicated. Dividing the $b$ phase space into high and low $|b|$ regions (to be defined below) one finds $\beta \to 0$ at high $|b|$ and therefore $Q(y, b)$ behaves like $\beta(b) \exp(\mu y)$, decreasing exponentially with increasing $|b|$. This region of $|b|$, in which $Q(y, b) \ll 2 \mu/r$, is characterized by the inequality $b^2 \gg R^2(y) = \mu y (R_0^2 + 4 \alpha' y)$, where $R_0$ is a typical radius defined by the exponential decrease of $\beta(b) \sim \exp(-b^2/R_0^2)$. At low values of $|b|$, inside the expanding radius $R(y)$, one finds for large $y$ that $Q \to 2\mu/r$, approaching its fixed point. (The picture is very similar to the one we had in sections 5.2 and 5.3.1.) The qualitative picture that emerges therefore for $Q(y, b)$ is a constant of height, $2\mu/r$, within a radius $R(y)$, while outside this disc $Q$ is small ($Q \ll 2\mu/r$). In the full quantum theory (section 7) in case $\mu > \mu_\infty$ and $\alpha' \neq 0$ a similar picture is obtained for $\sigma_T$; namely, $\sigma_T \sim (\ln s)^2$. Hence, we now have from (5.49) (after the $b$ integration was carried out):

$$\frac{1}{\sigma_T} \frac{\mathrm{d}\sigma}{\mathrm{d}y} \sim \frac{R(y) R^2(Y-y)}{\sigma^2(Y)}.$$  \hspace{1cm} (5.50)

Again, there are no conflicts with the inclusive sum rules. One also finds from eq. (5.50) that the asymptotic multiplicity at $\alpha' Y \gg R_0^2$ rises like $(\log s)^3$. The study of $k$ particles inclusive cross section results in $\langle n_k \rangle \sim \langle n \rangle^k$. Note that the requirement $\alpha' Y \gg R_0^2$ sets the scale of $Y$ at such high energies in which quantum corrections due to Pomeron loops become important. It was argued, however, that if RFT in the case $\alpha_0 > \alpha_\infty$ is treated as in section 7 below, which would show its consistency with the elementary requirements of $s$-channel unitarity, then the classical results in the present section will survive the quantum corrections [68].

6. Reggeon field theory on a lattice – path integral approach

The nature and origin of the universal behavior and its importance in understanding phase transitions was emphasized in section 1, along with its relevance to RFT. As explained in that section,
the universal behavior is independent of the detailed nature of the short range interactions and is characterized only by the dimensionality and the symmetries of the system. Hence, different models with similar symmetries, sharing a common universal behavior, are classified in the same universality class, where continuous models in space time, as well as discrete ones can be found. For example, much has been learned from the fact that the Ising model and the Ginzburg–Landau theory are in the same universality class [126]; they both have the same critical exponents and thus, the same power behavior at large distances at \( T = T_c \). It was therefore realized that a lattice model in the same universality class of the continuum Reggeon field theory can be very helpful in understanding many features of the theory. Certain techniques, developed in solid state physics, can be applied in order to deal with the analog spin system when one studies the nature of the phase transition in RFT, as well as for finding more accurate ways for calculating the scaling functions and the critical exponents of the theory.

The path integral formulation of RFT on a lattice, discussed in this section [40, 57], which though it yields several interesting results, is still difficult to handle in detailed numerical calculation [42, 84]. There is also a difficulty which will be discussed below with classifying the resulting lattice spin models in the same universality class of the continuum RFT. The approach to the problem which will be presented in section 7 gives a better insight into the issue of s-channel content of RFT, and the nature of its vacuum, and provides as well a very appealing scheme for numerical calculations (see section 7.3).

6.1. Formulation of the lattice theory and its implications on s-channel constraints in RFT

When formulating Reggeon field theory on a lattice we need a scheme to help us reduce the number of degrees of freedom at each lattice site. If the space time continuum is simply replaced by a discrete lattice, the degrees of freedom at each site are still infinite. Cardy and Sugar [57] used the path integral representation of the generating functional introduced in section 5 in order to formulate a lattice version of RFT. The number of degrees of freedom at each lattice site was then reduced to two, as dictated by the nature of the p, q phase space (discussed in section 5.2). The process of replacing the space time continuum by a discrete lattice that removes the ultraviolet divergencies should not in principle affect the interesting infrared behavior of the theory which we would like to study. This, however, as mentioned above, depends crucially on the presence of all the important symmetries of the original continuum theory in the lattice version of the theory.

Using relations similar to eqs. (2.11) and (2.12) [157, 92, 57], it can be shown that if \( \alpha_0 = \alpha_{0c} \) (eq. (1.24)), then \( r_0 \to \infty \) as \( g \to g_1 \). In this limit one also finds that \( \mu_c/r_0 \to \infty \) where \( \mu_c = \alpha_{0c} - 1 \). In section 5 we demonstrated how \( \mu \), and the unrenormalized triple Pomeron coupling, \( r_0 \) (which was equal to the renormalized one, \( r \)), had determined the fixed points of \( H(p, q) \) in eq. (5.21). Thus, ignoring for a moment the kinetic terms in \( \mathcal{L} \) we will discuss here the case where the two fixed points A and B in fig. 36 are very much apart from each other (\( \mu, r, \mu/r \to \infty \)). The functional integral in eq. (5.16) can be evaluated in this limit by the method of steepest descent. The contribution to the integral from the nearby region of the points \( (p, q) = (-2\mu/r, 0) \) and \( (0, -2\mu/r) \) is exponentially larger than that of the rest of the integration contours, as may be expected from the discussion in section 5. Denoting

\[
p = \chi + i\phi, \quad q = -\chi + i\phi
\]

and replacing the continuum \((\tau, x)\) space by a cubic lattice of unit cell spacing \( a_\tau \) and \( a_x \), the functional integral in eq. (5.16) has then the form
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\[ Z(g_1, g_2) = N \int \prod_i d\phi_i d\chi_i \exp \left[ -a_s a_x^2 \sum_i \mathcal{L}(\phi_i, \chi_i) \right]. \]  \hfill (6.2)

Written in terms of \( \phi_i \) and \( \chi_i \) (the fields at lattice site \( i \)), the main contributions to the integral come from the region near the points

\[ \{(p, q) = (-\frac{2\mu}{r}, 0), (0, -\frac{2\mu}{r})\} \rightarrow \{(\phi, \chi) = (\frac{\mu}{r}, -\frac{\mu}{r}), (\frac{\mu}{r}, \frac{\mu}{r})\}. \]  \hfill (6.3)

The variable \( s_i \) introduced by Cardy and Sugar \([57]\) plays here the role of a spin at lattice site \( i \), and takes the value \(+1\) or \(-1\). \( s_i \) is defined by

\[ \phi_i = i(\mu/r) + \phi' \]
\[ \chi_i = s_i(\mu/r) + \chi'. \]  \hfill (6.4)

The only important region of \( \phi' \) and \( \chi' \) is \( \phi', \chi' \approx O(\mu^{-1/2}) \); hence, the fields are effectively confined to the region near \( A \) and \( B \) of fig. 36. The gaussian integral obtained by keeping only second order terms in \( \phi' \) and \( \chi' \) can be evaluated in eq. (6.2), now including also the kinetic terms. The functional integral \( Z \) which plays the role of the partition function has finally the form

\[ Z = \sum_i \exp \left[ -K \sum_{(ij)} s_is_j \pm L \sum_{(ijk)} s_is_js_k \right]. \]  \hfill (6.5)

where \( K = (2\mu^2 a'/r^3)a_s \), \( L = \mu a'/r^2 \). \( i \) and \( j \) in the first term are nearest neighboring spin pairs in the space direction. \( i, j, \) and \( k \) in the second term are groups of three neighboring spins in space and time directions, where the signs \(+\) or \(-\) are given to different space time configurations. The invariance of \( \mathcal{L} \) in eq. (5.14) under the combined transformation \( q, p \to p, q \) and \( \tau \to -\tau \), corresponds to the transformation \( \chi \to -\chi \) and \( \tau \to -\tau \) in the spin model. These transformations in the lattice theory are a spin flip accompanied by a time reversal transformation (see eqs. (6.1) and (6.4)). The interactions in eq. (6.5) satisfy this symmetry as well as space reversal \( x \to -x \) symmetry. It was concluded \([57]\) that these symmetries of the spin model Lagrangian are sufficient to classify the model in the same universality class of the continuum Reggeon field theory, thus resulting in an identical critical behavior.

The above claim as to the classification of this spin model is, however, subject to controversy. It was noted \([41]\) that different transfer matrices are found in the \( 2+1 \) dimensional lattice theory presented by the Hamiltonian in eq. (6.5) and the RFT formulated on a two dimensional lattice in the impact parameter space, where the time dimension is left to remain continuous (section 7). This difference is due to a lack of a certain symmetry condition in the Cardy and Sugar model. The missing symmetry is related to an important condition on the persistence of the perturbative ground state for all values of the parameters in the lattice model of section 7. This difficulty is due to a lack of a certain symmetry condition in the Cardy and Sugar model. The missing symmetry is related to an important condition on the persistence of the perturbative ground state for all values of the parameters in the lattice model of section 7. The same problems are found also in the next section (6.2), where another formulation of RFT \([39, 40]\) with two spins at each lattice site is discussed. The assumption made by Cardy and Sugar was that in their model there exists a stable fixed point, but in view of the above difficulties which may prevent the classification of the model in the same universality class of the continuum RFT, one needs a somewhat stronger assumption \([41]\), namely, that it is possible to impose certain restriction on the different couplings in the model while performing the renormalization group transformation, to end up, however, at the fixed point suggested in ref. \([57]\). Such an assumption is very demanding and certainly is extremely difficult to check.

Given the spin model in eq. (6.5), techniques such as those developed for performing explicit renormalization group transformation for the Ising model \([123, 147]\), can be used. Sites are grouped
according to a chosen convention into cells. If, for example, cells of eight sites are defined, then group variables are translated into 2 cell variables and 2 internal cell variables, $\sigma$, which are summed. This defines a new Hamiltonian, $H'$, from

$$\exp\{-H'(s') - Ng\} = \sum_{\sigma_i} \exp\{-H(s_i)\}. \quad (6.6)$$

Even if the required symmetry properties are preserved in defining such a transformation, other possible spin interactions satisfying these symmetries are induced in addition to those in eq. (6.5). The relation between the old and new coupling constants is established by the repeated transformation from $H(KL \ldots)$ to $H'(K'L' \ldots)$. An explicit calculation of this type within a different formulation of RFT on the lattice will be discussed in section 7.3. No such calculations were done with the lattice theory presented above, due mainly to technical difficulties in handling a large number of variables in the process of performing the renormalization group transformation on eq. (6.5) [42]. Instead, as mentioned above, it was assumed [57] that a fixed point exists at $K = K^* = 0$ (or at some finite value, $K^* \neq 0$), $L = L^* \neq 0$, and $h = h^* = 0$, which is an added magnetic field, introduced in order to define correlation functions, using $Z(h)$. It is then found [57] that the exponent $1 - \zeta = z$ satisfies $z \leq 2$ as a result of the assumed stability of the fixed point in $K$. Furthermore, since the spin-spin correlation function is bound by unity, it restricts $-\gamma - z \leq 0$. Combining the two results, we have $-\gamma \leq 2$, which assures that the Froissart bound is not violated, and $-\gamma \leq z$, which assures that $\sigma_{t} > \sigma_{\text{f}}$ (eq. (2.41)), as well as $\sigma_{t} > \sigma_{\text{f}}$ (eq. (3.5)). Most of the numerical estimates of $-\gamma$ and $z$ in table 1 indeed satisfy the inequalities $-\gamma \leq 2$ and $-\gamma \leq z$, but here we have a more general argument saying that this is not a coincidence. Though the present formulation of RFT on a lattice had produced interesting results in the study of the s-channel content of the theory, we have to keep in mind the fact that the above results rely very heavily on strong assumptions concerning the classification of the model in the same universality class of the original continuum RFT and the existence of the fixed point.

Harms and Tan [115] had considered an interesting different formulation of RFT in terms of a single field $\chi$ defined in eq. (6.1). A certain gauge transformation is employed, separating the kinetic and potential pieces in the Hamiltonian and it is demonstrated that a smooth continuation from $1 - \alpha_0 = \Delta_0 > 0$ to $\Delta_0 < 0$ exists in their formulation. Using the path integral formulation described above, a classical spin model was derived on a lattice in rapidity and impact parameter space. The resulting interaction Hamiltonian differs from the one obtained in ref. [57]. It has, in addition to the terms shown in eq. (6.5), a nearest neighbor interaction in the $y$ direction and an additional next to nearest neighbors interaction term. Also, the coefficient $L$ determining the strength of the three spin coupling is different. In view of the restrictive assumptions needed to derive Cardy and Sugar’s inequalities for the critical exponents, it is not clear whether their results will persist here. The one component formulation of RFT [115] was used also to derive the quantum spin model of section 7.

6.2. Another path integral approach and the use of high temperature expansion in RFT

A similar formulation of RFT on a lattice in $x$ and $\tau$ space has been proposed also in a series of papers by Brower et al. [39, 40]. In their formulation of the theory, following standard solid state techniques, the transition to the discrete space-time lattice theory is made by defining two spins at each lattice site. Denoting $\phi_i$ and $\chi_i$ as the values of the fields (defined in eq. (6.4)) at lattice site defined by the vector $i$, which has two space dimensions and one time, the lattice theory Lagrangian is constructed, using the Lagrangian in eqs. (1.9) and (1.11), $\mathcal{L} = \Sigma \mathcal{L}_i$, where
In eq. (6.7) the infrared irrelevant term of strength, \( L \), was added to assure a smooth continuum limit for the ground state as the lattice space becomes very small [39]. This term does not violate any of the initial symmetries under which eqs. (1.9) and (1.11) are invariant [40]. The rest of the constants, \( E, K, M, G \), are simply related to the parameters \( \alpha_0, r_0, \Delta_0 \) and the space and time lattice spacings, \( a_0 \) and \( a_\tau \).

In order to restrict the spin variables to \( \pm 1 \) in a well-defined manner, a Gaussian integral transformation can be used. As in section 6.1, throughout this investigation special attention is paid to the symmetry properties of the new lattice Lagrangian, since as emphasized above, it is the symmetries involved that will determine whether the lattice theory is indeed in the same universality class of the original continuum model. Performing the Gaussian integral transformation by a method of steepest descent, it has been found that due to the shift to a stable vacuum implemented by \( \phi \rightarrow \phi' \), the equality between the two energy gaps, \( \Delta_\phi \) and \( \Delta_\chi \), associated with the fields \( \phi \) and \( \chi \) (eq. (6.4)), is not guaranteed. This violates the initial symmetries of (eq. (6.4)) and the theory with \( \Delta_\phi \neq \Delta_\chi \) will definitely not have the same infrared behavior as the theory determined by eqs. (1.9) and (1.11). On the other hand, if \( \Delta_\phi = \Delta_\chi \), one finds that the critical behavior is not sensitive to the new induced triple Pomeron interaction corresponding to the lattice system. To insure, however, that indeed \( \Delta_\phi = \Delta_\chi \), it is necessary to add new infrared relevant operators in the continuum theory such as

\[
(\psi^2 + \bar{\psi}^2), \quad (\nabla \psi \cdot \nabla \psi + \nabla \bar{\psi} \cdot \nabla \bar{\psi}).
\]

The lattice theory obtained now has other interaction terms in addition to those in eq. (6.7). In general, it is still possible that \( \Delta_\phi \neq \Delta_\chi \), but there are enough parameters to force the equality between the two gaps. The problem we faced in the Cardy and Sugar model in section 6.1 with the classification of their model in the same universality class as the continuum RFT, reappears here as well. It is most probable that we will end up with a less symmetric model than needed and new restrictions would have to be imposed in order to restore the symmetry. In trying to apply the standard renormalization group transformations in this model, one immediately realizes that even without new restrictions there are tremendous technical difficulties in carrying out such a program. In fact, having here two spins at each lattice site and a large number of different interactions had made the scheme even more complex for performing a numerical calculation [42].

Thus, we find here again, as in section 6.1, that other than several observations mentioned above, the path integral approach in formulating RFT on the lattice did not materialize into a convenient calculable scheme for its critical behavior. Many complications in the above scheme are due to the attempts to discretize the time dimension as well as the space dimension. (Another formulation of RFT on time-space lattice was treated in ref. [104].) In section 7 we will see that leaving the time \( y = -i\tau \) continuous, while having a lattice in the transverse space will result in a RFT on a lattice that would be much more easy to handle and consequently will produce many new results.

A single numerical attempt [84] however was carried out in order to find the critical exponents from the Lagrangian in eq. (6.7). In view of the discussion above it is not clear whether this result had anything to do with the equal gap \( \Delta_\phi = \Delta_\chi \) case in which we are really interested. In fact, adding the new terms may change considerably the estimates of the critical exponents calculated using eq. (6.7). This computation will be presented here anyway since the method can be used also in the lattice version in section 7, with much greater success [43].
The technique known from solid state physics as the high temperature expansion was used by Ellis and Savit [84] in order to estimate the numerical values of the critical exponents (see also on the same subject ref. [148]). This calculation method employs the expansion of statistical averaged quantities in powers of $T^{-1}$ whose role is played here by $K$ in eq. (6.7). For example, take

$$\langle v \rangle = \frac{\sum v e^{-\beta H}}{\sum e^{-\beta H}} = \frac{G}{T_c} \left( \frac{T - T_c}{T_c} \right)^{-\alpha}$$

(6.9)

then write

$$\langle v \rangle = \sum_{n=0}^{\infty} a_n T^{-n} = G \left( \frac{T}{T_c} \right)^{-\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n)}{\Gamma(n + 1) \Gamma(\alpha)} \left( \frac{1}{T_c} \right)^n$$

(6.10)

The ratio of successive terms in eq. (6.10) is:

$$R_n = \frac{a_n}{a_{n-1}} = T_c(1 + (\alpha - 1)/n).$$

(6.11)

Plotting $R_n$ as a function of $1/n$ enables the determination of $T_c$ as well as the critical exponent $\alpha$ defined above. Certainly, there is a need for several terms in the expansion to insure that the region of linearity of $R_n$ in $1/n$ is reached. In ref. [84] only three such terms were calculated (order $T^{-3}$). Higher order calculations are very tedious in view of the many possible graphs generated by eq. (6.7). The rough estimates of the critical exponents are given in table 1. The high temperature expansion in the quantum spin model of section 7 is discussed in section 7.3.

7. Reggeon field theory on a transverse space lattice and the analysis of the theory with a Pomeron intercept above its critical value

The formulation of Reggeon field theory on a lattice in rapidity and impact parameter space in section 6 had resulted in an analog classical spin system that suffered from several serious disadvantages which were discussed there. In this section the theory will be defined on a lattice in impact parameter space only, whereas the rapidity variable will be left continuous [20, 41, 115]. This Hamiltonian formulation of the theory results in an analog quantum spin model that gives much insight into several important issues: the nature of the phase transition, the structure of the vacua, and the ordered phase when $\alpha_0 > \alpha_{0c}$. It also provides a framework for numerical calculations such as renormalization group transformations [62] and high temperature expansion [43], to be discussed below.

7.1. Hamiltonian formulation of RFT on a lattice and the nature of the phase transition

The Pomeron field appearing in eqs. (1.9) and (1.11) is defined on a square lattice in impact parameter space at sites $x_i = an_i$, where $a$ is the lattice unit spacing and $n_i$ is a two dimensional ($D = 2$) vector, whose components are 0, ±1, ±2, etc.

$$\psi_n(\tau) = a^{Dz} \psi(\tau, x = an).$$

(7.1)

The Hamiltonian of the system is then defined on the lattice by replacing derivatives by finite differences and replacing integration by summation on lattice sites.
The summation on \( i \) is on nearest neighbors and we denoted \( \psi_n(\tau=0) = \psi_n^+ \) and \( \psi_n(0) = \psi_n^- \). (As explained in section 1, in general, \( \psi(\tau) \neq \psi^+(\tau) \), but the equality can exist however at a chosen \( \tau \); here we choose \( \tau = 0 \).) The commutation relation of \( \psi(\tau) \) and \( \psi^+(\tau) \) implies

\[
\{\psi(\tau, x), \psi^+(\tau', x')\} = \delta^D(x - x') \rightarrow [\psi_n, \psi_n^+] = \delta_{n,m}.
\]

The single site Hamiltonian \( H_{s0} \) in eq. (7.2) is identical to the Hamiltonian in eq. (4.3) with \( r_0 \) replaced by \( r_0 a^{-D/2} \). Its spectrum, discussed in section 4 [36, 119, 41], has a ground state at \( E = 0 \), and as \( \Delta_0 \rightarrow -\infty \) the first excited state has an energy \( E = \epsilon_i - \exp(-\Delta_0^2) \) given in eq. (4.12). All higher states have real energies of order \( |\Delta_0| \). It is possible now to follow similar technique employed in defining the quantum spin analog of the Ginzburg–Landau theory and also used in the calculations of its critical behavior [156]. Here we will study the theory as \( \Delta_0 \rightarrow -\infty \) and small \( a_0 \). In that regime only the ground state \( |0\rangle \) and the first excited state \( |1\rangle \) at each lattice site are expected to have significant matrix elements with those states of \( H \) in eq. (7.2), which are low in energy, and therefore important for the study of high \( s \) hadron scattering. In fact, one can choose sufficiently small \( r_0 \) and \( a_0 \) without affecting the universal behavior of the theory except for changing its critical temperature. The Hamiltonian in eq. (7.2) is then written in the truncated basis of \( |0\rangle_n \) and \( |1\rangle_n \). In order to find the matrix elements of \( \psi^+_n = \psi_n(0) \) and \( \psi^-_n = \psi_n^-(0) \) in this basis, the state \( |1\rangle \) used in section 4 has to be employed here as well. In the limit \( \Delta_0 \rightarrow -\infty \) one finds:

\[
\begin{align*}
\sigma_n^z &= -\frac{i}{2a^{D/2}} \left\langle \frac{r_0}{\Delta_0} \right|_n (i|\psi_n(0)|j)_n = \frac{i}{2}(1 - \sigma_+^z) - \sigma_+^z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\sigma_n^y &= -\frac{i}{2a^{D/2}} \left\langle \frac{r_0}{\Delta_0} \right|_n (i|\psi_n^+(0)|j)_n = \frac{i}{2}(1 - \sigma_+^z) + \sigma_+^z = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\end{align*}
\] (7.4a,b)

where the 2x2 matrix representations of the fields in the \( |0\rangle, |1\rangle \) basis are written in terms of Pauli matrices. (Note that the notations here are the same as in ref. [20], and that they differ by a \( \pi/2 \) rotation around the \( z \) axis from those in [62] and in eq. (3.21) of ref. [41].) The non-hermiticity of the original single site Hamiltonian \( H_{s0} \) in eq. (7.2) is reflected here in the non-hermitian representations for \( \sigma_n^x \) and \( \sigma_n^y \), though \( \psi(y=0) = \psi^+(0) \). In fact, as will be discussed below, in deriving eqs. (7.4a,b) one has to distinguish between left and right hand eigenstates of \( H \), which are not conjugates of each other [20], and are related by a certain metric operator.

The single site part of the Hamiltonian in eq. (7.2) in the truncated basis \( |0\rangle \) and \( |1\rangle \) is written as a simple sum on the number operators at each site since its spectrum in this basis is simply \( E = 0, \epsilon_i \), where \( \epsilon_i \) is given in eq. (4.12) (with \( r_0 \rightarrow r_0 a^{-D/2} \))

\[
\sum_n H_{n0} = \sum_n \left[ \Delta_0 \psi_n^+ \psi_n + \frac{i}{2a^{D/2}} \left\langle \frac{r_0}{\Delta_0} \right|_n \psi_n^+ (\psi_n^+ + \psi_n) \psi_n \right] \rightarrow \sum_n \frac{\epsilon_i}{2} (1 - \sigma_+^z).
\] (7.5)

Adding also the intersite interaction term one finally obtains from eq. (7.2) (using the identity \( \sigma_n^z \sigma_n^z = 0 \) which is however true only in the truncated basis. Alternatively, in order to avoid this relation, one can add the appropriate term to \( \Delta_0 \), see, for example, in ref. [115]):
where \( \hat{e} \) is the unit vector and

\[
J = 4a_\theta^2 \left( \frac{\Delta_0/a_\theta}{r_0} \right)^2 a^{D-2} = \epsilon_1/X. \tag{7.6b}
\]

By varying the relative weights of \( 1 - \sigma_\perp \) and \( \sigma_\perp \) in eq. (7.4a,b) at fixed \( J \) and \( \epsilon_1 \to 0 \), the quantum spin Hamiltonian in eq. (7.6a) is related to familiar solid state spin models. The case \( \rho_2 (1 - \sigma_z) \gg \rho_1 \sigma_\perp \) reduces the system to an Ising model Hamiltonian and \( \rho_2 (1 - \sigma_z) \ll \rho_1 \sigma_\perp \) gives an \( xy \) model \((J \to -J)\).

In a certain sense the weights in eq. (7.4a,b) \((\rho_1 = \rho_2 = 1)\) place the quantum spin RFT in between these two models [98]. This observation is useful when renormalization transformations are performed [62] and the critical surfaces in parameter space is studied.

\( X \) in eq. (7.6b) will play the role of the temperature, since only the ratio between the coefficients of \( H_0 \) and \( H' \) control the spectrum of \( H \), and the phase transition that is found in the system occurs therefore at a certain value of \( X = X_0 \). Then, through the expression of \( \epsilon_1 = \epsilon_1 (\alpha_\theta', r_0, \Delta_0, a) \) in eq. (4.12) (replace there \( r_0 \to r_0 a^{D-2} \)) the critical temperature \( X_0 \) determines the dependence of \( \Delta_{0c} \) on the other parameters. We have

\[
\Delta_{0c}^2 = \frac{r_0}{2} a^{D-2} \ln \left( \frac{r_0}{\sqrt{\pi} \alpha_\theta' X_0} \right). \tag{7.7}
\]

The ultraviolet cutoff is determined here by the lattice spacing \( a \) rather than by a cutoff in angular momentum space, \( E \), as in eq. (1.24). It is desirable, of course, to be able at the end of the calculations with a fixed \( a \) to express all the final measurable quantities independently of \( a \), so that as we then go smoothly back to the continuum limit as \( a \to 0 \) the physical picture developed below for the nature of the phase transition and the ordered phase will not be affected.

A similar method with a two level truncation was applied to the Ginzburg–Landau Hamiltonian on a lattice in \( D \) space dimensions [156] given by

\[
H = \sum_n \left[ \frac{1}{2} \Pi_n + \lambda_0 \left( \phi_n^2 + \frac{m_0^2}{4\lambda_0} \right)^2 + C^2 \sum_i (\phi_{n+i} - \phi_n)^2 \right]. \tag{7.8}
\]

This method reduces the theory to a quantum \( D \) dimensional Ising model in a transverse magnetic field

\[
H = \sum_n \left[ \frac{\epsilon}{2} (1 - \sigma_z^n) + \delta \sum_i (1 - \sigma_z^{n+i} \sigma_z^i) \right]. \tag{7.9}
\]

Since this model at \( D = 1 \) is equivalent to the asymmetric classical Ising model in \( d = D + 1 = 2 \) dimensions [159], it is concluded that the two dimensional (one space, one time) real \( \phi^4 \) theory is in the same universality class of the two-dimensional Ising model, and it is believed that this is true for higher dimensions as well [126]. The asymmetric Ising model at \( d = 2 \) has the same critical behavior as the symmetric one, namely, the critical exponents of the Onsager's solution [129, 150]. Knowing the exact critical behavior at \( d = 2 \) of the Ising model makes it possible to check the qualities of approximation schemes at \( D = 1 \) for the quantum Hamiltonian in eq. (7.9) [156]. In the case of RFT on a lattice the situation is more complicated. As explained in section 6, the symmetry properties of the Cardy and Sugar model [57] raised some difficulties with the classification of this model in the same universality class of Reggeon field theory [41]. Thus we do not have a reliable classical spin model in \( d = D + 1 \) dimensions, which is in the same universality class of the \( D \) dimensional quantum spin
Reggeon field theory Hamiltonian in eq. (7.6). Therefore, evaluating the merits of the approximation schemes applied in studying the critical behavior of the quantum spin Hamiltonian in eq. (7.6), we have to rely often on the experience gathered from investigations of the $\phi^4$ theory in eq. (7.8) [63, 156]. There are however several fundamental differences between the $\phi^4$ theory and Reggeon field theory, in particular as to the nature of the vacua and phase transition, which we will discuss below.

In the single site problem in eq. (7.8) one recognizes the Schrödinger one-dimensional unharmonic oscillator problem. The two lowest energy states at the site correspond to the symmetric and antisymmetric combination of the wave function in the two wells at $\phi = \pm \sqrt{-m^2_0/4\lambda_0}$ of the potential $V(\phi) = (\phi^2 + m^2_0/4\lambda_0)^2$. It is these two single site states $|0\rangle_n$ and $|1\rangle_n$ in which the Hamiltonian in eq. (7.9) was truncated; they are separated by a small energy gap $\epsilon \sim \exp(m^2_0/\lambda_0)$ due to the tunnel effect and become degenerate as $m^2_0 \rightarrow -\infty$. For the full Hamiltonian in eq. (7.8) one finds two degenerate ground states, $|0\rangle$ and $|1\rangle$, at a finite negative $m^2_0$. However, $|1\rangle$ is not the unperturbed ground state and there are non-zero matrix elements of $\phi$ between $|0\rangle$ and $|1\rangle$. If the new ground states are redefined $|g_\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$ one finds that in the infinite lattice case there are no non-vanishing matrix elements of $\phi$ between the states $|g_\pm\rangle$. Namely,

$$\langle g_\pm|\phi|g_\pm\rangle = \frac{1}{2}\left(\langle 0|\phi|0\rangle \pm \langle 1|\phi|0\rangle \pm \langle 0|\phi|1\rangle - \langle 1|\phi|1\rangle\right) \rightarrow 0$$

is vanishing as $\exp[-N^2]$ where $N$ is the number of lattice sites. (For the infinite lattice $\langle 0|\phi|0\rangle = \langle 1|\phi|1\rangle = 0$ and $\langle 0|\phi|1\rangle = \langle 1|\phi|0\rangle \neq 0$.) The field $\phi$ has non-vanishing expectation values in the new ground states $|g_\pm\rangle$:

$$\pm \sigma = \langle g_\pm|\phi|g_\pm\rangle = \frac{1}{2}\left(\langle 0|\phi|0\rangle \pm \langle 1|\phi|0\rangle \pm \langle 0|\phi|1\rangle \pm \langle 1|\phi|1\rangle\right) = \pm \langle 1|\phi|0\rangle.$$  

The original $\phi \rightarrow -\phi$ symmetry is therefore spontaneously broken and $\sigma$ is the order parameter which behaves like $\sigma \sim (X - X_c)^\theta$ where $X = \epsilon/\delta$. (For example, like the magnetization in a ferromagnet as $T \rightarrow T_c$.)

We will find that Reggeon field theory's behavior is entirely different from that of the $\phi^4$ theory described above. In view of the crucial role played by the symmetry of the system in determining the character of the phase transition, it is important to investigate this aspect first. The symmetry property of $H$ in eq. (7.2), playing here an important role, is sometimes called the "pseudo-hermiticity" of the Hamiltonian. As discussed in section 1, we are dealing with a non-hermitian $H$ and therefore $\bar{\psi}(\tau, x) \neq \psi^*(\tau, x)$. There exists however an operator $U$ which relates $\psi^*$ and $\bar{\psi}$ and demonstrates a certain symmetry of the system [20, 41]

$$U = U^{-1} = U^* = \exp \left[ -i\pi \int d^2x \bar{\psi}(0, x)\psi(0, x) \right]$$

$$U\psi(\tau, x)U^{-1} = -\bar{\psi}^*(-\tau, x), \quad U\bar{\psi}(\tau, x)U^{-1} = -\psi^*(\tau, x)$$

$$UHU^{-1} = H^*.$$  

The symmetry under this transformation corresponds to the symmetry between $G^{nm}$ and $G^{mn}$, namely, the exchange of the projectile and the target and is certainly not a symmetry we would like to break. It had been unjustifiably broken [11] in an early attempt to understand the ordered phase of RFT. In the quantum spin model defined by eqs. (7.6) and (7.4a,b), $U$ is replaced by $M$ where $M$ is given by

$$M = M^{-1} = M^* = \prod \sigma^z_n$$
and the "pseudo-hermiticity" of $H$ is displayed by (see eqs. (7.4a,b) and (7.6))

$$M\sigma_\phi^+M^{-1} = \sigma_\phi^+,$$

$$M\sigma_\phi^-M^{-1} = \sigma_\phi^-$$

(7.12b)

$$M\hat{H}M^{-1} = \hat{H}^\dagger.$$

This is not a symmetry in the usual sense but it plays here a role analogous to the $\phi \to -\phi$ symmetry in $\phi^4$. It is satisfied by eqs. (7.4a,b) and (7.6a) and we find that if $\Psi_i$ is a right hand eigenstate of $H$ ($H\Psi_i = \epsilon_i \Psi_i$) then $\Psi_i'M$ is a left hand eigenstate of $H$ ($\Psi_i'M^\dagger = \Psi_i'H$). The operator $M$ acts as a metric operator and is used in computing matrix elements of local operators in the basis spanned by the eigenstate of $H$ (e.g., in computing eq. (7.4a,b) [41]).

At $X > X_c$, where $X$ is defined in eq. (7.6b), the vacuum is the perturbative vacuum given by the collective state (we discuss now the collective states of the spin model)

$$\Phi_0 = \prod_n |0\rangle_n = \prod_n \left( \frac{1}{0} \right)_n.$$  

(7.13)

Namely, at all sites the ground state $|0\rangle_n$ satisfies $H^{(0)}_n|0\rangle_n = 0$ and is annihilated by $\psi_n (\sigma_\phi^+|0\rangle_n = 0)$. The fact that RFT has a behavior different than the $\phi^4$ theory has, can be seen when eqs. (7.4a,b) are compared with the $\phi^4$ case. In RFT $|0\rangle$ are not the new ground states of the ordered phase at $X < X_c$; consequently, the analogous equations to (7.10a,b) are different, and therefore the nature of the change in the vacua at the critical point is different. In fact here $\Phi_0$ itself continues to be a ground state of the system even at $X < X_c$, but now it is degenerate with a new negative norm collective state $\Psi_0$. A variational calculation of $(\Psi_0, H_0 \Psi_0)/(\Psi_0, \Psi_0)$ subject to the condition $(\Psi_0, \Phi_0) = 0$ in the limit $\Delta_0 \to -\infty$ gives [20]

$$\Psi_0 = \Phi_0 - \Phi_\sigma = \Phi_0 - \prod_n X_\sigma^n$$

(7.14a)

where

$$X_\sigma^n = \left( \begin{array}{c} 1 \\ -\sigma \end{array} \right).$$

(7.14b)

In this approximation $\sigma = 1 - X$ for $X < X_c$, but in general, $\sigma(X)$, which is proportional here to the matrix elements that connect the two degenerate vacua, $\Phi_0$ and $\Psi_0$ (see below) plays the role of the order parameter $\sigma(X) \sim (X_c - X)$. We have now a vanishing vacuum expectation value of the field in the original vacuum $\Phi_0$:

$$\langle \Phi_0|\psi_\sigma|\Phi_0 \rangle = \langle \Phi_0|\psi^*_\sigma|\Phi_0 \rangle = 0$$

(7.15a)

and the following non-diagonal matrix elements are non-vanishing

$$\langle \Phi_0|\psi_\sigma|\Psi_0 \rangle = \langle \Psi_0|\psi^*_\sigma|\Phi_0 \rangle = -2ia^{\partial\partial_x} |\Delta_0| |r_0| \sigma(x).$$

(7.15b)

The expectation values of $\psi$ and $\psi^*$ in the new vacuum $\Psi_0$ are also non-vanishing. At $X$ near $X_c$ one finds:

$$\langle \Psi_0|\psi_\sigma|\Psi_0 \rangle = \langle \Psi_0|\psi^*_\sigma|\Psi_0 \rangle = -2ia^{\partial\partial_x} |\Delta_0| |r_0| \sigma^2(x).$$

(7.15c)

In obtaining eqs. (7.15a,b,c) the metric operator $M$ was employed in order to compute correctly the bra vectors. Thus, the non-diagonal matrix elements behave here in a way different from that in the $\phi^4$ case, where they vanish as negative exponents of the volume or the number of lattice sites in the system. Here $\psi$ and $\psi^*$ have finite matrix elements between the two degenerate vacua at $X < X_c$. 


and unlike those in eq. (7.10b), the vacuum expectation values of $\psi$ and $\bar{\psi}^*$ vanish in the ground state $\Phi_0$. The expectation values of the fields in the collective vacuum states $\Phi_0$ and $\Psi_0$ in eqs. (7.15a,b,c) show also that the $\psi \leftrightarrow \bar{\psi}^*$ symmetry, which corresponds to the exchange of the beam and the target, is preserved.

The implications of the vacua structure of the ordered phase can be now studied. Defining the coupling of the Pomeron to the external particles as in eq. (5.15) and expressing the Green’s functions in eq. (1.12) in terms of the lattice Pomeron fields $\psi_n$ and $\bar{\psi}_n$, one finally defines the $S$ matrix as [20]:

$$S(Y, B) = \langle \Phi_0 \left| \exp \left\{ ia^{D2} \sum_n \beta_1(n) \psi_n \right\} e^{-YH} \exp \left\{ ia^{D2} \sum_n \beta_2(n) \bar{\psi}_n \right\} \right| \Phi_0 \rangle.$$  (7.16)

The expression should however be considered with some caution [41, 62]. It was assumed above, as in section 5, that the creation and annihilation of the Pomerons at the particle lines take place in an uncorrelated manner. This is an approximation which, as discussed in section 5, is more suitable for nuclei than for hadrons. We cannot take sources which are localized at a single impact parameter since a power $I$ of the field $(\psi_1, \bar{\psi}_1)$, calculated in the truncated basis, is not proportional to $(\sigma)^I$ and thus the exponentiation is impossible. If we assure however that the sources are very extended in impact parameter space the expression in eq. (7.16) is more acceptable. The alternative, of course, is to focus our attention directly on the Green’s function $G^{n.m}$.

In the limit of fixed $B$ and $Y \rightarrow \infty$, a complete set of eigenstate of $H$ introduced between the operators in eq. (7.16) is dominated by the low lying states. The contribution of the two degenerate ground states, $\Phi_0$ and $\Psi_0$ is [20, 62]:

$$S(Y, B) = 1 - T(Y, B) = 1 - \left[ 1 - \exp \left\{ -2\sigma |\Delta_0| r_0 \sum_n a^D \beta_1(n) \right\} \right] \times \left[ 1 - \exp \left\{ -2\sigma |\Delta_0| r_0 \sum_n a^D \beta_2(n) \right\} \right].$$  (7.17)

In the limited $B$ region in which this expression is valid, it gives a grey disc. The entire contribution to the $T$ matrix in eq. (7.17) comes from the non-vanishing non-diagonal matrix elements of $\psi$ and $\bar{\psi}^*$ between $\Phi_0$ and $\Psi_0$ in eq. (7.15b). Thus, it is the appearance of the new ground state $\Psi_0$ in the ordered phase, $\alpha_0 > \alpha_0$, that gives rise to a constant $T$ matrix at fixed $b$. The first non-leading term contributing to $T(Y, B)$ is proportional to $\exp(-\omega_0 Y)$, where $\omega_0 > 0$ is the gap between the ground state energy and the first excited state which is a collective excitation mode. The continuum spectrum of these modes given by the frequencies $\omega(k)$ was calculated [20] in the mean field approximation. Other non-leading terms are attributed to contributions of higher one-site excited levels; they are however important only outside a disc of a radius growing like $B = \ln s$ in impact parameter space [20]. If this disc is accepted as the region of validity of eq. (7.17), we end up with a total cross section increasing like $(\log s)^2$. We will see, however, in section 7.2 that it is not the above particular disc that determines the nature of the expanding disc of the total cross section.

From eq. (7.17) one finds that the individual Green’s functions $G^{n.m}$ are also proportional to the non-diagonal matrix elements in eq. (7.15b) which play here a major role

$$G^{n.m}(Y, B) = \left( \frac{2\Delta_0}{r_0} \right)^{n+m-2} G^{1,1}(Y, B)$$  (7.18a)

$$G^{1,1}(Y, B) = (2\Delta_0/r_0)^2 \sigma^2.$$  (7.18b)

Thus, unlike the Cheng Wu [64, 65] model that eikonaiizes $G^{1,1} \sim s^\epsilon (\epsilon > 0)$ and results in $\sigma_T \sim (\log s)^2$,
here $G^{1,1}$ already satisfies by itself the Froissart bound. Note also that eq. (7.18a), being derived from eqs. (7.16) and (7.17) is subject to the same eikonal type approximation.

As emphasized in this section, there are basic differences between the nature of the ordered phase in $\phi^4$ theory and the ordered phase in RFT. It is instructive to compare the analysis described above of the $\alpha_o > \alpha_{oc}$ phase in RFT with the study of ref. [11], where an attempt was made to follow a close analogy with the $\phi^4$ theory. We will then see how one gets into serious difficulties by following such an analogy too closely. Defining $\eta_0 = \alpha_{oc} - \alpha_o$ one finds, as in section 5, that the classical potential $U_c = \eta_0 \bar{\psi} \psi + \frac{1}{2} i r_0 \bar{\psi}(\bar{\psi} + \psi) \psi$ has for $\eta_0 < 0$ ($\alpha_o > \alpha_{oc}$) two acceptable stationary points at $\psi = v$, $\bar{\psi} = \bar{v}$, where

\begin{align*}
  v = 0, & \quad \bar{v} = 2i\eta_0/r_0 \\
  v = 2i\eta_0/r_0, & \quad \bar{v} = 0.
\end{align*}

(7.19)

The classical potential written in terms of the shifted fields $\psi' = \psi - v$, $\bar{\psi}' = \bar{\psi} - \bar{v}$ is:

\begin{equation}
U_c(\psi', \bar{\psi}') = \delta_o \bar{\psi}' \psi' + \frac{1}{2} i r_0 \bar{\psi}' \bar{\psi}' + \frac{1}{2} i r_0 \bar{\psi}' \bar{\psi}' + \frac{1}{2} r_0 (\bar{\psi}' \psi' + \bar{\psi}' \psi')
\end{equation}

(7.20)

where $\delta_o = \eta_0 + i r_0 (v + \bar{v})$, and using eq. (7.19), we have $\delta_o = -\eta_0 > 0$. Since in a given order in $r_0$ there is only a finite number of $\psi'^2$ or $\bar{\psi}'^2$ vertices, these terms in eq. (7.20) can be included in the interaction Lagrangian.

In order to obtain partial wave amplitudes without unwanted delta functions in momentum space, one couples the scattering particles to correlation functions $\bar{G}^{n,m}$, which are the Green's functions of the displaced fields $\psi'$ and $\bar{\psi}'$. The bare propagator of the $\psi'$ fields in the case $\eta_0 < 0$ (eq. (7.20)) is:

\begin{equation}
\bar{G}^{1,1}_0(E, k^2) = i(E - \alpha' k^2 - (\eta_0) + i\epsilon)^{-1}.
\end{equation}

(7.21)

The renormalized one particle irreducible proper vertex functions $\bar{\Gamma}^{n,m}$ can be treated in the case $\eta_0 \neq 0$ in the same way as $\Gamma^{n,m}$ were treated in sections 1 and 2. We have here, however, an additional parameter, $\delta = Z_0 \delta_0$, which has dimensions of energy $E$ and will enter into the renormalization group equation. For the small $E, k^2$ and small $\delta$ one finds:

\begin{equation}
\bar{\Gamma}^{1,1}(E, k^2, g, \alpha', \delta, E_N) \xrightarrow{\delta \to 0} E_N C_0 (\delta/E_N)^{1-\gamma(1-\kappa)} \Phi(\rho_1 \rho_2)
\end{equation}

(7.22)

where

\begin{align*}
  \rho_1 &= C_1 \frac{E}{E_N} \left( \frac{\delta}{E_N} \right)^{-1(1-\kappa)} \\
  \rho_2 &= C_2 \frac{\alpha' k^2}{E_N} \left( \frac{\delta}{E_N} \right)^{-2(1-\kappa)}
\end{align*}

(7.23)

and $\kappa$ is a new critical exponent. When the limit $\delta \to 0$ is taken for fixed $\rho_1, \rho_2$, one finally obtains:

\begin{equation}
\bar{\Gamma}^{1,1}(E, k^2, g, \alpha', \delta, E_N) \xrightarrow{\delta \to 0} E_N C_0 \left( \frac{E}{E_N} \right)^{1-\gamma} \Phi \left( C_1 \frac{\delta}{E_N} \left( \frac{E}{E_N} \right)^{-(1-\kappa)}, C_2 \frac{\alpha' k^2}{E_N} \left( \frac{E}{E_N} \right)^{-\kappa} \right).
\end{equation}

(7.24)

Continuing from $\eta_0 > 0$ to $\eta_0 < 0$ (namely, $\alpha_o < \alpha_{oc}$ to $\alpha_o > \alpha_{oc}$) constant sources $J \psi + \bar{J} \bar{\psi}$ were used in the same manner an external magnetic field is used in choosing one of the two ground states in a ferromagnet and enables a continuation through $T = T_c$. The result in eqs. (7.22–7.24) is the generalization of the scaling result in eq. (1.20) for $\delta \neq 0$ and it reduces to it in the limit $\delta \to 0$. The new critical
exponent, $\kappa$, as well as the scaling functions, were evaluated in the $\epsilon$ expansion [11]. The physical intercept gap $\Delta$ and slope $\Delta'$ for small $\delta$ are ($\kappa = \epsilon/12$):

\[
\Delta = 1 - \alpha_r(0) \frac{\delta}{\frac{2\epsilon}{\pi} - 1/3} \left\{ 1 - \frac{\epsilon}{6} \left[ \frac{1}{2} - (1 - \text{sgn}(\eta_0)) \ln 2 \right] \right\} \quad (7.25a)
\]

\[
\alpha_r'(0) \frac{\delta}{\frac{2\epsilon}{\pi} - 1/3} \left\{ 1 + \frac{\epsilon}{6} (1 - \text{sgn}(\eta_0)) (\frac{3}{2} - 2 \ln 2) \right\} \quad (7.25b)
\]

$\delta$ is always positive and proportional to $|\eta_0|$. The intercept gap $\Delta = 1 - \alpha_r(0)$ is zero only if $\eta_0 = 0$ ($\alpha_0 = \alpha_{0c}$). Otherwise, it is always positive and the renormalized Pomeron intercept was therefore claimed to be below one in the case $\alpha_0 > \alpha_{0c}$ as in the case $\alpha_0 < \alpha_{0c}$ [11].

There is however a very severe problem with the above analysis. As long as $\alpha_0 \leq \alpha_{0c}$ eqs. (7.22–7.24) and similar relations for $\tilde{\Gamma}^{n,m}$ (which equal in that case to $\Gamma^{n,m}$) as well as eq. (7.25) are all correct and no difficulty arises. However, there is a problem in the case $\alpha_0 > \alpha_{0c}$ where, upon turning off the sources, one of the two possibilities in eq. (7.19) is chosen and therefore in eq. (7.20) (see also [85]) $v = 0, \bar{v} \neq 0$ or $v \neq 0$ and $\bar{v} = 0$, choosing one of the two cases, say, $v = \langle \psi \rangle \neq 0$, one finds that there is a $\frac{1}{2} r_0 v \bar{v}^* = \delta_0 \bar{v}^2$ term added to the interaction. Such a term will appear in the $\tilde{G}^{1,2}$ correlation function as shown in fig. 42b. However, it does not appear in the diagram in which A and B are interchanged (fig. 42a), making A the source for the two Pomeron. Similarly, it is now clear that $G^{n,m} \neq G^{m,n}$. Since the amplitude for scattering particle A (projectile) on particle B (target) is given by

\[
M_{AB} \sim \sum_{n,m=1}^{\infty} g_n^A G_{n,m}^A G_{m}^B
\]

one finds that the interchange of projectile and target will give in general different result, $M_{AB} \neq M_{BA}$, and thus, breaking the Lorentz invariance. All this indicates again that the nature of the phase transition in RFT is entirely different from the one in $\phi^4$ theory as discussed throughout this section. This is something one cannot see by analyzing only the form of the classical potential, as done in ref. [11] (see also ref. [172]).

7.2. The asymptotic behavior of $\sigma_T$ in the ordered phase ($\alpha_0 > \alpha_{0c}$), the expanding disc and $s$-channel unitarity

The mechanism that gives rise to the expanding disc (discussed in the previous section) was shown to be associated with a vacuum instability property of the Reggeon field theory [21]. Its detailed rigorous analysis was carried out however only in $D = 1$ transverse dimension. The vacuum instability that will be discussed below has a special character. It is not due to the existence of lower lying states that forces one to redefine the vacuum as in $\phi^4$ theory with $m^2 < 0$. Here, we find that applying a field operator on the vacuum $\Phi_0$ causes a disturbance that propagates in impact parameter space at a constant velocity and fills the whole space as $Y \rightarrow \infty$. The time evolution of this disturbance that takes the system from one degenerate vacuum to the other gives rise to the expanding grey disc in eq. (7.17).
Consider first the case $\Delta_0 \to -\infty$, namely, in eq. (7.6) $\epsilon_i \sim \exp(-\Delta_0^2)$ is set to zero; and define a family of "box states", $\xi^{l,m}$ (fig. 43a)

$$\xi^{l,m} = \prod_{n < 1} \chi^0_n \prod_{l} \chi^n_l \prod_{n > m} \chi^n_n. \tag{7.26}$$

$\chi^n_n$ has been defined in eq. (7.14b). Using eqs. (7.4a,b) one finds:

$$\begin{align*}
\sigma^n_x \chi^n_0 &= \sigma^n_x \chi^n_1 = 0 \\
\sigma^n_x \chi^n_i &= \chi^n_i \\
\sigma^n_x \chi^n_0 &= \chi^n_0 - \chi^n_i.
\end{align*} \tag{7.27}$$

Note also that if $\epsilon_i = 0$ in eq. (7.6), then $H\Phi = H\Pi_{m} \chi^n_m = 0$; hence, $\Phi_0$ and $\Phi_1$ (defined in eq. (7.14a)) are the two degenerate ground states of the system. From eq. (7.27) we have

$$\begin{align*}
\sigma^n_x \xi^{l,m} &= 0 & l &\leq n &\leq m \\
\sigma^n_x \xi^{l,m} &= 0 & n &< l \quad \text{or} \quad n &> m \\
\sigma^n_x \xi^{l,m} &= \xi^{l,m} & l &\leq n &\leq m.
\end{align*} \tag{7.28}$$

We see therefore that only the edge of the box state will be affected by terms of the type $\sigma^{n}_{\phi} \sigma^{n+1}_{\phi}$, where $i$ stands for a nearest neighbor. Thus using the Hamiltonian in eq. (7.6) with $\epsilon_i = 0$, one finds that the evolution of the box state is given by

$$-\xi^{l,m} = H_0 \xi^{l,m} = J[2\xi^{l,m} - \xi^{l-1,m} - \xi^{l+1,m}]. \tag{7.29}$$

The general solution of this equation in the continuum is ($b = la$ and $b' = ma$)

$$e^{-H_0 \xi} \xi^{l,m} = \xi(b, b', y) = \xi(b - Jay, b' + Jay, 0). \tag{7.30}$$

The box state $\xi$ expands with a velocity $v = Ja$. Note that in one transverse dimension (in which eq. (7.30) was obtained), $v$ is independent of $a$; namely, we have (eq. (7.6b)) $v = Ja = 4a^2 \Delta_0^2 r_0^2$. The expanding box state is directly related to the grey disc behavior of the renormalized Pomeron

$$G^{1,1}(ib, jb, Y) = \left(\frac{2\Delta_0}{r_0}\right)^2 \langle \Phi_0 \sigma^{i}_{\phi} e^{-H_0} \sigma^{j}_{\phi} | \Phi_0 \rangle. \tag{7.31a}$$

$\sigma^{i}_{\phi} | \Phi_0 \rangle$ creates a "box" state, $\xi^{i,j}$, as seen from eqs. (7.4a), (7.13) and (7.27), which then expands as indicated in eq. (7.30). Finally, by using also eq. (7.20) one finds:

![Fig. 43. The vacuum instability represented in various families of states ("box" (a), "step" (b,c), "hole" (d)).](image)
\[ G^{1,1}(ib, jb, Y) = \left( \frac{2\Delta_0}{r_0} \right)^2 \langle \Phi_0 | \sigma_j \xi(jb - JaY, jb + JaY) \rangle \]
\[ = \left( \frac{2\Delta_0}{r_0} \right)^2 \Theta(JaY - |i - j|b). \] (7.31b)

Similarly, for the \( S \) matrix the expression of eq. (7.17) is obtained here as well, and is multiplied however by \( \Theta(JaY - B) \). Thus in one transverse dimension one finds \( \sigma_j \sim \log s \). It is important to note that the radius of the disc \( JaY = (4\alpha_0\Delta_0^2/r_0^2)Y \) is not the one which determines the region of validity of the two level truncation approximation at each site (section 7.1 and ref. [20]), but is directly related to the evolution of the states \( \xi^{l,m} \) [21]. (For a classical picture of the disc see ref. [70].)

The \( \{ \} \) plane structure of the two point function in eq. (7.31b) (note, \( D = 1 \) and \( \epsilon_i = 0 \)) is:

\[ G^{1,1}(E, k) = \frac{1}{E} \left( \frac{1}{E - iJa} + \frac{1}{E + iJa} \right) = \frac{1}{E^2 + J^2 a^2 k^2}. \] (7.32)

The evolution of the disturbances of the type similar to the one discussed above can be demonstrated also in the family of states in figs. 43b,c,d. Their evolution exhibits the same instability of the vacuum \( \Phi_0 \), which forces the system from a disturbed \( \Phi_0 \) to the \( \Phi_1 \) state. It is easy to see that the form of the nearest neighbors interaction in the Hamiltonian of eq. (7.6) (with \( \sigma_\phi \) and \( \sigma_\delta \) given in eq. (7.4a,b)) will always result in an equation of motion of the type similar to eq. (7.29) with a tendency to fill up the space with \( \chi_1 \) states at each site. The soliton like solutions are represented here by the non diagonal matrix elements of \( \psi_n \) and \( \psi_n^* \) between the left and right step states of fig. 43b,c. Again, one realizes [21] how the non-diagonal matrix elements of the fields play here the role of the expectation value of the field in the \( \phi^4 \) theory.

In the case \( \epsilon_i \neq 0 \) the states \( \Phi_0 \) and \( \Phi_1 \) are no longer degenerate vacua. However, we have seen in section 7.1 that at \( X < X_c \) the two degenerate vacua \( \Phi_0 \) and \( \Psi_0 \) exist. The picture with \( \epsilon_i \neq 0 \) was studied in the small \( X \) region [21], where, again, one finds an expanding grey disc for the \( T \) matrix. The opacity and the velocity decrease as \( X \) increases towards \( X_c \). At \( X > X_c \) mean field approximation shows a Regge pole-like behavior \( (X_c = 1 \) in this approximation). In two transverse dimensions \( (D = 2) \) the intercept is at

\[ a = 1 + 8\alpha_0(\Delta_0|r_0|^2)(1 - X) < 1 \] (7.33a)

and the slope is

\[ a' = 4\alpha_0(\Delta_0|r_0|^2)a^2. \] (7.33b)

As discussed in section 7.1, it is desirable to end up with measurable results that are independent of the lattice spacing so that there will be a smooth continuum limit. In the case of the disordered phase \( (\alpha_0 < \alpha_{oc}) \), \( a' \) in eq. (7.33b) cannot be kept fixed in the limit \( a \to 0 \) unless the bare parameters are varied as well. It is not clear, however, how this is implemented in the quantum spin system [21]. In fact, note that also in the ordered phase \( (\alpha_0 > \alpha_{oc}) \) one faces a problem with the \( a \to 0 \) limit; though at \( D = 1 \) the velocity \( v = Ja \) is finite, this is not the case at \( D = 2 \) as can be seen in eq. (7.6b). The issue of the smooth transition needs further studies (see the very recent development on the continuum limit \( a \to 0 \) in the ordered phase in ref. [36.1] – discussed in section 8).

The ordered phase region \( X < X_c \) \( (\alpha_0 > \alpha_{oc}) \) was discussed above in one transverse dimension \( (D = 1) \). The two-dimensional case, where our main interest lies, has been treated on a less rigorous level. The operation with \( \psi \) on the vacuum \( \Phi_0 \) produces a "bundle state" whose radius, even in the case \( \epsilon_i \neq 0 \), was conjectured [21] to increase with a constant velocity, \( Ja \), thus giving again an
expanding grey disc for the $T$ matrix and $\sigma_{\gamma} \sim (\log s)^2$ at $D = 2$. The detailed bubble states behavior as well as the $a \to 0$ limit at $D = 2$ is not yet fully understood.

An analysis of the theory with $\alpha_0 > \alpha_{\text{oc}}$ was carried out in ref. [97] directly at $D = 2$ in the continuum theory. Using the representation for the $G_{\gamma}^{1,1}$ Green's function in ref. [95] (see section 2) it was shown that if the critical exponent that corresponds to the renormalized intercept satisfies a certain inequality, the cross section is dominated by a cut with a branch point at $E = 0$ and behaves like $(\log s)^{r}$, where $r = 1.1.2$. The analytic continuation was performed in the representation obtained in the first loop approximation only. However, as shown in section 2, it seems that certain very appealing features (for example, those regarding the angular distribution) appear in the Green’s function already at that approximation level. Further studies are certainly needed also here in order to clarify, in particular, the relation of this work to ref. [21] and the nature of the phase transition discussed in section 7.1.

At this point, recalling the discussion in section 3, it will be very interesting to proceed and study the properties of production amplitudes in the case $\alpha_0 > \alpha_{\text{oc}}$. Although it seems that no violations of $s$-channel unitarity occur at the elastic scattering amplitude level, and all elastic partial waves in eq. (7.17) are bounded, it is important to verify that indeed, no problem will arise from inelastic diffractive production amplitudes as well. It has been shown (in section 3) that difficulties such as those encountered when trying to implement $s$-channel unitarity constraints in theories with weak cuts do not appear in the scaling solution of Reggeon field theory ($\alpha_0 = \alpha_{\text{oc}}$). In the case $\alpha_0 > \alpha_{\text{oc}}$ discussed here, several constraints concerning diffractive production were analyzed at different levels of approximations. The analysis of the inclusive cross section in the central region and in the triple Regge region with arbitrary $\alpha_0 > 1$ had been carried out at the classical level only [68, 142], and was presented in section 5.3. The analysis of multiparticle diffractive production cross sections presented below was carried out in detail in one transverse dimensions ($D = 1$) for the quantum spin model and studied in a semiclassical approximation at $D = 2$ [137].

The $n$ particles diffractive production amplitude (fig. 44) is given in RFT by

$$G_n^{\psi}(B, Y, b_j, y_j) = \left\langle 0 \left| \left( \prod_{j=1}^{n} \psi(\mathbf{y}_j) \psi(\mathbf{b}_j) \right) \left( \psi(0, 0) \right) \right| 0 \right\rangle. \quad (7.34)$$

In one transverse dimension ($D = 1$) and the case $\epsilon_j = 0$ in eq. (7.6) an explicit solution can be found. The mass insertion operator on the lattice, $(\psi \psi)_n$ is calculated in the two level truncated basis $|0\rangle$ and $|1\rangle$ discussed in section 7.1, in which $\psi_n$ and $\psi_n^1$ were calculated (eq. (7.4a,b)). The operator $Z_n$ in this basis is given by [137]:

$$Z_n = \frac{\rho^2}{8\epsilon_1 \Delta_0} (\psi \psi)_n = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \quad (7.35)$$

and one finds

$$Z_n \chi_0^n = 0, \quad Z_n \chi_1^n = \chi_1^n - \chi_0^n \quad (7.36)$$

and therefore (eq. (7.26))

Fig. 44. A general $n$ particle diffractive production amplitude.
\[ Z_{n}^{\xi_{l,m}} = 0 \quad \text{if} \quad n < l \quad \text{or} \quad n > m \]
\[ Z_{n}^{\xi_{l,m}} = \xi_{l,m} - \eta_{l,n,m} \quad \text{if} \quad l < n < m \] (7.37)

where \( \eta_{l,n,m} \) is a box state with a "hole" at \( n \).

The Green's function in eq. (7.34) is given essentially by a product of the matrix elements of the mass insertion operator \( Z_{n} \) in the intermediate states of the type shown in fig. 44. The properties of \( Z_{n} \) displayed in eqs. (7.28) and (7.29) show that matrix elements of \( Z_{n} \) between these states will have contributions only from their expanding edge. Thus, the production amplitude is peripheral and the only important interaction region in impact parameter space is near the edge of the expanding discs \( b_{i} = v_{j} = J_{i}y_{j} \). The amplitude vanishes exponentially outside this region causing therefore an effective reduction of the available phase space. The \( y_{i} \) distribution is governed by the absorptive character of the Reggeon field theory that we already have met in section 3.1 in the case of \( \alpha_{0} = \alpha_{oc} \), which induces a clustering of the produced particles in the phase space region near the target and near the projectile. One then finds [137] that the effect of this dynamical reduction of the rapidity and impact parameter phase spaces results in an asymptotic constant \( n \) particles diffractive production cross section. The sum of all the partial cross sections gives also a constant diffractive inelastic cross section

\[ \hat{\sigma}_{in}(E) = \sum_{n=1}^{\infty} \hat{\sigma}_{n}(E) E^{-1-i-j} \left( \frac{1}{2J - g^2} \right). \] (7.38)

This result, which is correct in the range in which the coupling constant for producing particles satisfies \( g^2 < 2J \), shows that a Finkelstein–Kajantie type of problem that was shown [27] (section 3.1) to disappear in the case \( \alpha_{0} = \alpha_{oc} \), does not appear in the case \( \alpha_{0} > \alpha_{oc} \) either. One also learns from the result in eq. (7.38) that here as well we expect the non-diffractive production process to dominate the total inelastic cross section.

In the case \( e_{i} \neq 0 \) and \( D = 2 \), which is much more complicated, it was argued on a less rigorous level [137] that the quantitative picture discussed above remains unchanged.

The issue of the s-channel content of RFT looks very promising and there are many indications that indeed RFT does satisfy s-channel unitarity constraints. This was demonstrated by the expanding grey disc type behavior of \( \sigma_{T} \) and the analysis of \( \sigma_{n} \) described above. There are however several important problems left open for more studies and new ideas. First, a detailed demonstration of a disc type behavior was derived in \( D = 1 \) only, and a more rigorous proof at \( D = 2 \) as well is desirable, where the analysis is somewhat more difficult. The continuous limit of taking the lattice spacing to zero confronts us with several problems. Finally, there is a need for an analysis within the quantum spin model of other s-channel unitarity constraints for the inclusive cross sections. This was done at the tree graph level in the continuous theory in section 3 and it needs to be extended possibly in the lattice framework to the quantum level. It is very desirable, in order to analyze the effect of the disc mechanism on inclusive cross sections, to formulate the cut Reggeon field theory discussed in section 3.4 on the transverse lattice. The quantum spin cut Reggeon field theory can be a very useful tool for further analyzing the s-channel unitarity content of the theory in the \( \alpha_{0} > \alpha_{oc} \) region. (See in the Note added in proof, ref. [10P].)

7.3. Calculations of the critical exponents in the quantum spin model

The quantum spin Reggeon field theory formulated in section 7.1 provides an analog spin system in the same universality class of the continuum RFT, which is however a much simpler system to analyze and perform numerical calculations with. In fact, one can use here the arsenal developed for
the treatment of phase transitions in statistical mechanics. Of particular interest are the successful numerical methods for calculating the critical exponents employed in solid state physics, whose merits are tested in some cases vis a vis exact solutions or experimentally in many other cases, an advantage we do not presently share in RFT. Two such calculations will be presented here: first [62], the use of approximate renormalization group transformation [123, 134, 147] of the type used recently for the quantum Ising model in a transverse field [156]. Second, a promising high temperature expansion in a work still in progress [43] will be briefly discussed.

The parameter space for the renormalization group transformation is defined in eqs. (7.6) and (7.4a,b). One can let the system choose also different weights for \( \sigma_+ \) and \( 1 - \sigma_+ \) in eq. (7.4a,b), thus replacing \( \sigma_\phi \) and \( \sigma_\phi^* \) in eq. (7.6) (absorbing \( J \) into the \( \sigma_\phi \)'s) by:

\[
\begin{align*}
\sigma_\phi^n &= \frac{1}{2} \rho_2 (1 - \sigma_+^n) + \rho_1 \sigma_+^n \\
\sigma_\phi^* &= \frac{1}{2} \rho_2 (1 - \sigma_+^n) - \rho_1 \sigma_+^n.
\end{align*}
\]  

(7.39)

(We continue to use the notations of eqs. (7.4a,b) and (7.6), which differ by a \( \pi/2 \) rotation around the \( z \) axis relative to ref. [62].) In one transverse dimension \( (D = 1) \) the approximate renormalization group transformation is performed as follows [62]: each two neighbor sites (fig. 45) are combined into a cell; however, since each site has two states and a cell has four, the cell state is approximated by a truncation of the cell Hamiltonian at each step of the transformation. This is done by taking into account only the two lowest states in each cell. The site Hamiltonian, \( H \), is then written in terms of cell operators

\[
H = \sum_n \frac{\epsilon_1}{2} (1 - \sigma_+^n) + \sum_{n,i} \sigma_\phi^{n+i} \sigma_\phi^{n+i} + \sigma_\phi^{n+i} \sigma_\phi^{n+i}
\]

\[
\rightarrow H' = \sum_n \frac{\epsilon_1'}{2} (1 - \sigma_+^n) + \sum_{N,i} \sigma_\phi^{N+i} \sigma_\phi^{N+i} + \sigma_\phi^{N+i} \sigma_\phi^{N+i}
\]

(7.40)

where \( \sigma_\phi^n, \sigma_\phi^* \) are given by eq. (7.39) and the cell operators \( \sigma_\phi^N \) and \( \sigma_\phi^N \) have the same form of eq. (7.39) with \( \rho_1, \rho_2 \) replaced by new parameters, \( \rho_1', \rho_2' \). By requiring that the cell operators have the same matrix element as the site operators in the cell basis, one determines the relation between the \( \rho_1, \rho_2 \) and \( \rho_1', \rho_2' \). The new energy gap \( \delta'(\epsilon_1', \rho_1') \) between the ground state and the excited state is calculated directly from the single cell piece in the Hamiltonian \( H \). Thus, performing repeatedly such transformations, a path is defined in parameter space by the relations between the new \( \{ \epsilon_1', \rho_1', \rho_2' \} \) and the old \( \{ \epsilon_1, \rho_1, \rho_2 \} \) parameters at each step. Each path is defined by its initial conditions given by the values of \( \epsilon_1, \rho_1, \rho_2 \). Starting for example at \( \rho_1 = \rho_2 = \sqrt{J} \) does not imply that \( \rho_1' = \rho_2' \), since there is no symmetry in the system to force this equality. As mentioned also in section 7.1, varying \( \rho_1/\rho_2 \) teaches us about the relation in parameter space between the quantum spin RFT, the Ising model, and an \( xy \) model with a positive sign for the coupling [98].

The structure obtained [62] for the parameter space is a typical example of a system that has a second order phase transition at some \( X = X_c \) [126, 134]. There are three fixed points so that paths with temperatures \( X > X_c (\alpha_0 < \alpha_{0c}) \) and paths with \( X < X_c (\alpha_0 > \alpha_{0c}) \) end up at different stable fixed points. Starting with \( X = X_c (\alpha_0 = \alpha_{0c}) \), the path ends at the third fixed point which is stable with

![Fig. 45. Renormalization group transformation by combining two sites at each step into a cell and thus \( H \rightarrow H' \) in eq. (7.40).](image-url)
respect to $\rho^2/\rho_0^2$ and unstable with respect to $\varepsilon_1/\rho_0^2$. Using now the standard technique relating the behavior in the vicinity of the fixed point to the exponents $\gamma, z$ defined in section 1, and adding the exponent $\kappa$, defined by $\Delta \sim (X - X_c)^{1/(1 - \kappa)}$, where $\Delta$ is the renormalized Pomeron's energy gap, one finds [62] (at $D = 1$):

$$-\gamma = 0.27, \quad z = 2.5, \quad \kappa = 0.22$$

(7.41)

and the critical temperature is $X_c = 1.15$.

Consider now a single cell matrix element of $\psi_n$ or $\tilde{\psi}_n$ in the ordered phase. In this phase, as the cell becomes larger, $\epsilon_1$ approaches zero and there is no energy gap between the ground state and the first excited state. In this region the matrix elements of the field operators in eq. (7.15) are proportional to $\sigma(X)$, which in section 7.1 were seen to play the role of the order parameter. $\sigma(X)$ appears in the asymptotic form of the $S$ matrix (eq. (7.17)) and in Green’s function $G^{1,1}$ (eq. (7.18)). It has the form $\sigma(X) \sim (X_c - X)^\beta$ and $\beta$ was calculated in ref. [62] (in $D = 1$, $\beta = 0.64$).

Cardy [62] had calculated the critical exponents also in two dimensions. The approximate renormalization group transformation was defined on a triangular and square lattice. The triangular lattice gives presumably unreliable results considering the fact that at each step one truncates the $2^2$ states to two. For the square lattice the possibility that was considered was combining two site cells [62]. The results for the critical exponents ($D = 2$) are (for the square lattice and triangular lattice, respectively):

$$-\gamma = 1.4, 2.; \quad z = 4.8, 8.; \quad \kappa = 0.06, -2.7; \quad \beta = 1.06, 1.0,$$

and the critical temperature $X_c = 2.87, 3.10$.

The analog calculation in solid state physics [156] was carried out with the Hamiltonian of eq. (7.9) of the quantum Ising model in a transverse field in one dimension, which as explained in section 7.1 is equivalent to the classical Ising model in two dimensions. One can then compare the results of the approximate renormalization group transformation calculations with the exact Onsager’s solution. Indeed, one finds that whereas one of the critical exponents come out pretty good: 0.9 versus 1., the other exponents are 3.6 compared to 2 in the exact calculation and 0.37 versus 1/8. Returning back to Cardy’s work [62], one does not expect therefore to gain more than an estimate from this approximation. In particular, the exponent $z$ is certainly too large. Certain refinement of the calculation in order to compensate for the fact that the system is truncated at each step was suggested [98]. The above calculation demonstrates however that we are at a position which enables us to use solid state physics techniques on the quantum spin RFT Hamiltonian of eq. (7.6) and try to improve our understanding of the system. Hopefully one would like to be able to determine the critical exponents accurately enough so that if the critical theory ($\alpha_0 = \alpha_{oc}$) is indeed relevant to high energy experiments in the present or the near future (see sections 2.4 and 2.5) RFT will then have unique and accurate predictions.

Another attempt, using high temperature expansion, to obtain accurate values for the critical exponent, is in progress [43] and the few preliminary results seem to be promising. It will be discussed here only briefly since the final results are not yet available.* The high temperature expansion, explained in section 6.2, serves in solid state physics as one of the most accurate methods for calculating critical exponents and critical temperatures [155]. The first attempt to use it with the path integral version of RFT in eqs. (6.7) and (6.8) had encountered serious difficulties due to the complicated spin model it treated. In fact, the results of ref. [84] are based on an expansion up to $T^{-3}$ which is far from being of a high enough order as solid state physics calculations seem to demand [155]. The RFT quantum spin Hamiltonian of eq. (7.6) is far more simpler. (In fact, calculations up to order $T^{-9}$ in one dimension and up to $T^{-5}$ in two dimensions for the Green’s function $G^{1,1}(n, y)$ were carried out even without the use of a computer. We were able to carry out our expansion to order $T^{-20}$ using a short computer program.) Examining, for example, the quantity

*See Note added in proof and the last line in table 1 section 1.
\[ R_n = \frac{c_n}{a_{n-1}} - u_c^{-1}(1 + \frac{a_n}{n}) \]

Fig. 46. Preliminary results of the high temperature expansion at \( D = 1 \). \( a_n/a_{n-1} \) is perfectly linear at \( n > 3 \). The slope and intercept can be found very accurately.

\[ \chi(X) = \sum_n G^{1,1}(n, E = 0, X) \]

one finds that in addition to the singularity on the positive \( X \) axis at \( X = X_c \) there are other singularities in the left half plane. After a conformal mapping of the temperature plane \( U = U(X) \) \((U \to 0 \text{ as } X \to \infty)\) which does not change the critical exponent of \( \chi \) but carries away the irrelevant singularities, a power series in \( U \) for \( \chi(U) \) is obtained. We expect \( \chi(U) \) to behave like \( (U_{c} - U)^{-\alpha} \) (where \( \alpha = \alpha(\gamma, z, \kappa) \)) and therefore \( R_n \) in eq. (6.11) to be linear in \( 1/n \) for large \( n \). Indeed, \( R_n \) was found to have a perfect linear behavior (see, for example, fig. 46), and the intercept and slope can be determined very accurately. Figure 46 looks as good as a typical high temperature expansion in ref. [155]. We also find that the slope and the intercept of \( R_n \) are independent of small changes in the parameter of the conformal mapping \( U = U(X) \). From fig. 46 and similar pictures we expect to obtain the critical exponents and the critical temperature with a very good accuracy. All this looks very promising, but apparently, it will not be completed by the time writing this report will end. It is hoped that no difficulties will be encountered and a satisfying and accurate calculation of the exponents will be reached.

8. More recent developments in the analysis of the theory with \( \alpha_0 > \alpha_{oc} \)

Upon the conclusion of this review two new works, treating the theory with a Pomeron above its critical value, came to our attention and will be briefly discussed below (Bronzan and Sugar – UCSB preprint, in preparation [36.1] and White [169]). The authors of the first paper suggest an interesting modification of the theory into an hermitian theory, which has several advantages and also reproduces many of the results of refs. [20] and [21], discussed in section 7. In ref. [169], the difficulties with \( t \)-channel unitarity in the results of section 7 are discussed and a very different conclusion is reached as to the asymptotic behavior of the theory in the ordered phase \( (\alpha_0 > \alpha_{oc}) \).
Bronzan and Sugar [36.1] considered the theory with an interaction Hamiltonian which, in addition to the triple Pomeron interaction in eq. (1.11), has also a four Pomeron interaction of the type \( \frac{1}{4} \lambda_0 \psi^2 \psi^2 \). Requiring that the Pomeron's intercept will retain a special value \( \Delta_0 = \Delta_{0m} = -\frac{r_0^2}{\lambda_0} < 0 \), one can construct, in addition to the ground state \( |0\rangle \) (\( H|0\rangle = 0 \), \( \psi|0\rangle = 0 \)) a degenerate vacuum which is a coherent state of the bare Pomeron \( (H|\phi\rangle = 0 \), \( \psi|\phi\rangle = 2i(\Delta_0/r_0)|\phi\rangle \)) and is given by

\[
|\phi\rangle = \exp \left( i2 \frac{\Delta_{0m}}{r_0} \int_{-\infty}^{\infty} d^D x \psi^*(x) \right) |0\rangle.
\]  

(8.1)

The ordered phase can be further explored now by performing a similarity (but not unitary) transformation:

\[
S\psi S^{-1} = \psi, \quad S\psi^* S^{-1} = \psi^* + i2(\Delta_{0m}/r_0)
\]

(8.2a)

with

\[
S = \exp \left( -i2 \frac{\Delta_{0m}}{r_0} \int_{-\infty}^{\infty} d^D x \psi(x) \right).
\]

(8.2b)

This shift of the field \( \psi^* \) transforms the non-hermitian RFT Hamiltonian \( H \) (including the \( \psi^2 \psi^2 \) term) into an hermitian Hamiltonian

\[
H' = S^{-1}HS = \int_{-\infty}^{\infty} d^D x \left[ \alpha_0 \psi^* \psi - \frac{r_0^2}{4\Delta_{0m}} \psi^* \left( \psi^* + i2 \frac{\Delta_{0m}}{r_0} \right) \right] \left( \psi - i2 \frac{\Delta_{0m}}{r_0} \right). \]

(8.3)

\( H' \) is positive semi-definite with a degenerate zero energy vacua chosen as \( |0\rangle \) and \( |1'\rangle \), given by:

\[
|1'\rangle = \left[ \exp \left( \frac{4\Delta_{0m}^2}{r_0^2} V \right) - 1 \right]^{-1/2} (|\phi\rangle - |0\rangle)
\]

(8.4)

where \( V \) is the volume of the impact parameter space and the normalization is such that \( \langle 0|0\rangle = \langle 1'|1'\rangle = 1 \). It was argued that there exists a finite gap between these vacua and the first excited states, and the states \( |0\rangle \) and \( |1'\rangle \) remain degenerate in the limit \( V \to \infty \), when \( \Delta_0 \) is varied in a finite range of values around \( \Delta_{0m} \) provided that the Green's functions are indeed analytic at \( \Delta_0 = \Delta_{0m} \). Thus, showing a typical behavior of an ordered phase in a system that has a second order phase transition.*

The quantum spin model in this hermitian theory is formulated in a way similar to the one in the theory with \( \lambda_0 = 0 \) in section 7.1. Here, however, one finds several advantages. The spectrum of the single site Hamiltonian is:

\[
E_n = \frac{1}{2} \lambda_0 a^{-D} n(n - 1) + n(\Delta_0 - \Delta_{0m}) + O(a^D).
\]

(8.5)

Thus, if \( \Delta_0 = \Delta_{0m} \) is fixed, and \( a \to 0 \), then \( E_0 = 0 \), \( E_1 = 0 \) and all other energies \( E_n \sim a^{-D} \). The hermitian quantum spin model is then formulated by using the two lowest states at each site as in refs. [20, 41]. Here, however, the justification for such a formulation of a spin model does not break down in the limit \( \Delta_0 \), \( r_0 \) fixed and \( a \to 0 \) as happens in the spin model with a triple Pomeron coupling only (see section 7.1), since in this limit \( E_0 = E_1 = 0 \) is preserved in eq. (8.5). This property is important for the analysis of the \( t \)-channel unitarity content of the theory [36.1].

The Green's function \( G^{11}(x, y) \) (eq. (1.12)) can be written either in terms of the original Hamiltonian \( H \) or, after performing the similarity transformation into the hermitian theory, in terms of \( H' \),

*The analyticity at \( \Delta_0 = \Delta_{0m} \) is however questionable.
\[ G^{1,1}(x, y) = \langle 0 | \psi(x) e^{-H_S \psi}(0)|0 \rangle = \langle \phi | \psi(x) e^{-H_S \psi}(0)|0 \rangle. \]  

(8.6)

Note that in calculating \( G^{1,1}(x, y) \) with the hermitian \( H' \), the vacuum \( |0 \rangle \) is replaced by \( |0 \rangle |S^{-1} = \langle \phi | \). Here one can already see the important difference between a conventional hermitian field theory, where we would have had \( |0 \rangle \) instead of \( \langle \phi | \) and the RFT where one starts with a non-hermitian \( H \). In the former case (see section 7.1) matrix elements of local operator between states, which are excitations of the two distinct vacua, \( |0 \rangle \) and \( |1' \rangle \), vanish like the exponential of the quantization volume and therefore the second vacuum will not contribute in eq. (8.6) if \( \langle \phi | \) is replaced by \( |0 \rangle \). In such a case \( G^{1,1} \) vanishes as \( y \to \infty \) and the renormalized Pomeron singularity is below one in the ordered phase. Here, however, as seen from eq. (8.1) \( \langle \phi | \psi(x)|1' \rangle \) diverges as \( V \to \infty \) and therefore \( \langle \phi | \psi(x)|1' \rangle (1'|\psi(x)|0 \rangle \to \text{const.} = 4\Delta_0/r_0^2 \). Thus, as in section 7.1, the non-vanishing non-diagonal matrix elements of the fields imply

\[ G^{1,1}(x, y) \xrightarrow{y \to \infty, x \text{ fixed}} \text{const.} \]

which means that the highest singularity of \( G^{1,1}(E) \) in the ordered phase is at \( E = 0 \).

The contributions of other intermediate states were calculated in one transverse momentum \( (D = 1) \) in the quantum spin model and in the continuous theory. In the former case, when \( H' \) operates on a box state (see section 7.2) it affects only its edge and therefore maps box states into themselves. Hence in eq. (8.6) (note \( a_j|0 \rangle \sim |jj \rangle \), the same as in eq. (7.31a)):

\[ e^{-H' a_j|0 \rangle} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 C(x_1, x_2, y)|x_1x_2 \rangle \]

(8.7)

where \( |x_1x_2 \rangle \) denotes a box state of the type of fig. 43a, which is an eigenstate of \( \psi(x) \) with eigenvalue \( 2i \Delta_0/r_0 \) if \( x_1 < x < x_2 \) and eigenvalue zero otherwise (see eq. (7.28)). \( C(x_1, x_2, y) \) can be solved [36.1]:

\[ C(x_1, x_2, y) = \frac{\Delta_0^2 (x_2 - x_1)}{32 \pi \Delta_0 |a_j| y} \exp \left[ - \frac{8 \alpha_0 \Delta_0^4}{r_0^4} y - \frac{x_1^2 + x_2^2}{4 \alpha_0 y} \right]. \]

(8.8)

Finally, convoluting eq. (8.7) with

\[ \langle \phi | a_j |x_1x_2 \rangle \sim \theta(x_1 - x_1)\theta(x_2 - x_2) \exp(2\Delta_0^2 (x_1 - x_2)/r_0^2) \]

(8.9)

gives \( G^{1,1}(x, y) \). It behaves approximately as an expanding disc in agreement with eq. (7.31b) in section 7.2. Adding also the non-leading piece gives a cross section that behaves like (recall \( D = 1 \) and \( \alpha_0 > \alpha_0^c \)):

\[ \sigma_T \sim c_0 \log s + c_1 (\log s)^{1/2} \]

(8.10)

where the \( (\log s)^{1/2} \) term is a contribution of a transition region of width \( (\sqrt{\alpha_0} y) \).

The spin model calculations are equivalent to calculations in the continuum model, where only the contribution of the truncated Hilbert space of single box and kink states are considered. A detailed calculation of the contribution of other states to eq. (8.6) (e.g., states with two, three etc. boxes) can further reveal the nature of the approximation involved in this calculation [44]. It may also clarify the apparent lack of Reggeon unitarity in the above result. The leading pole singularity found in \( G^{1,1} \) (eq. (8.6)) is indeed at \( j = 1 \) (at \( t = 0 \)), but the usual accompanying cuts with branch points at \( j = 1 \) required by Reggeon unitarity are absent here [36.1]. This is a severe problem which needs further studies, since as explained in section 1, Reggeon unitarity relations provided the main motivation for the development of RFT.
An interesting study of Reggeon field theory with a $\lambda_0 (\overline{\psi} \psi)^2$ interaction was carried out also in ref. [100] (with $r_0 = 0$, however). Using the inverse scattering method it was concluded that in the classical limit, the $S$ matrix is unbounded. The inclusion of the triple Pomeron interaction, as well as quantizing the theory, may very well change this result as we have seen above.

The apparent lack of manifest $t$-channel unitarity in the results of the quantum spin model for the ordered phase was criticized by White [169]. If indeed the total cross section at $D = 2$ rises like $(\log s)^2$, the leading singularity is in contradiction with $t$-channel unitarity constraints. It may well be that this is a result of a complicated $j$-plane singularity structure that for $\alpha_0 > \alpha_{oc}$ eliminates all singularities with $\alpha_R(0) > 1$ off the physical sheet [169]. This requires, however, the existence of at least an additional cut in the angular momentum plane that is not generated through the pole and cuts in the Reggeon unitarity relations (see section 1). Hence, the Reggeon unitarity relations do not control the singularity structure and one cannot demonstrate exact $t$-channel unitarity [108, 167].

In the ordered phase it is noted that in addition to the two fixed points in eq. (7.19), there exists also the classical solution of the type shown in eq. (5.27), which joins the two asymmetric fixed points. This symmetric instanton-like solution is argued to play a major role in the ordered phase [169]. In terms of the quantum spin model discussed in section 7, White’s suggestion implies a redefinition of the $S$ matrix in eq. (7.16), where the original vacuum $\Phi_0$ is replaced by the new instanton vacuum. This prevents any zero eigenstates contributions in the intermediate states as in eq. (7.17) which gave a constant $T$ matrix. Thus, the $T$ matrix will vanish now like a power of $s$ and the leading singularity in the ordered phase is below one. This reproduces in fact the results of ref. [11], discussed in section 7.1, but the disastrous effect of the asymmetry between the two fixed points in eq. (7.19) is eliminated here by using instead the symmetric instanton vacuum that connects these two points.

If indeed the correct definition which preserves the $t$-channel unitarity properties of RFT is given by replacing $\Phi_0$ in eq. (7.16) by the instanton vacuum and $\alpha_R(0)$ is indeed below one, then the critical theory ($\alpha_0 = \alpha_{oc}$) is the only one that predicts a factorized rising asymptotic cross section as a power of $\log s$, which seems to be supported by experiments. We then have to understand, maybe outside RFT, why the underlying strong interaction theory implies a special constraint between the parameters that represent the bare Pomeron, so that $\alpha_0 = \alpha_{oc}$. It is not clear, however, whether White’s suggestion is the only remedy for the difficulties with $t$-channel unitarity in the ordered phase. The inclusion of multi-box states may very well clarify this important issue [44] that certainly needs further studies.

9. Conclusions

We will conclude this report with a short discussion of the present status of RFT as seen from two perspectives: First, looking back at what has already been achieved, the main results mentioned in the previous sections will be briefly summarized and a few comments added. Second, those issues and problems which were left open and need further studies and examination, will be mentioned and briefly discussed.

In view of the very attractive $t$-channel unitarity properties of RFT, discussed in the Introduction, it is interesting to note the extensive recent progress achieved in Reggeon field theory, especially in understanding its $s$-channel content, which had thus turned it into a very appealing framework for the analysis of hadron scattering at high energies. Though much progress has been made recently, elucidating the nature of the ordered phase and the phase transition, several important problems still
exist in the case $\alpha_0 > \alpha_{0c}$. The case of the critical Pomeron ($\alpha_0 = \alpha_{0c}$) is however very well understood and long standing problems and inconsistencies in the analysis of the complex angular momentum structure of scattering amplitudes were shown not to be shared by RFT.

For the case $\alpha_0 = \alpha_{0c}$ very useful techniques (presented in section 2) were developed for calculating the scaling functions and the approach to scaling. The results, especially for the structure of the forward diffractive peak, are certainly encouraging. The relation between the perturbative approach and the critical asymptotic theory were clarified, though the numerical evaluation of the transition energy between the two relevant energy regimes, as well as the value of $\alpha_{0c}$, are not completely settled yet. The studies of the approach to scaling encourages further analysis of the phenomenological implications of RFT.

A substantial portion of this review was dedicated to the discussion of the issue of the s-channel content of RFT, which has recently gained much attention. One of the approaches to this important problem discussed in section 3 was the studies of the particular constraints imposed by s-channel unitarity on certain scattering amplitudes, checking whether these constraints are indeed satisfied in RFT. These studies at $\alpha_0 = \alpha_{0c}$ had shown that the absorptive effect of the Pomeron interaction prevents such disastrous results which led to the Finkelstein–Kajantie problem and the decoupling of the Pomeron singularity in theories with weak cuts. Also, the central region in the inclusive cross section does not seem to contradict s-channel unitarity demands. A very interesting implementation of s-channel unitarity requirements in RFT was presented in section 3; the resulting cut Reggeon field theory describing s-channel discontinuities of scattering amplitudes is in agreement with the AGK cutting rules and presents a very powerful tool for studying s-channel issues.

The analysis of RFT at zero transverse dimensions and its classical tree graph approximation gave much insight into the spectrum of the theory and its $p$, $q$ phase space structure. These developments, discussed in sections 4 and 5, were extremely useful in later studies and in building the lattice quantum spin analog model that belongs to the same universality class of the continuum RFT. The exact solution at $D = 0$ serves also as a laboratory, though a limited one, for checking the merits of various approximation schemes. The phenomenological implications of the classical approximation have several interesting consequences. The application of RFT ideas to hadron nucleus scattering seems to be the most promising one, although at present there is much more to be done, and the situation is far from being satisfactory. It is still a matter of controversy whether the $A^{1/3}$ enhancement of the triple Pomeron coupling indeed makes it relevant to present experiments.

The issue of s-channel unitarity in the ordered phase ($\alpha_0 > \alpha_{0c}$) was studied by many authors. The most fruitful approach to the problem proves to be through the quantum spin analog of RFT. Previous formulations of RFT on the lattice, which resulted in classical spin models on a lattice in the transverse and rapidity space, had produced several interesting results regarding the s-channel properties of the critical theory ($\alpha_0 = \alpha_{0c}$), and a very limited framework for conducting numerical calculations. It also exhibited several disadvantages discussed in section 6: in particular, there are doubts as to whether these models can be classified in the same universality class of the continuum RFT. The quantum spin model presented in section 7 has been much more successful in revealing the nature of the phase transition and the analysis of the s-channel constraints in the ordered phase. The important difference between the type of phase transition occurring in RFT and those in other conventional hermitian theories is now very well understood. It was certainly reassuring to find out that the constraints imposed by s-channel unitarity, which were analyzed to date are indeed satisfied in the ordered phase. Although the detailed analysis of the asymptotic behavior was carried out in one transverse dimension, there are reasons to believe that conflicts with s-channel unitarity will be
avoided in two dimensions as well. The quantum spin model, which also provided a framework for the
detailed numerical calculations of the critical behavior, will most probably continue to be in the future
an extremely useful technical tool for analyzing several of the problems which are listed below.

A different perspective on the status of RFT is attained by looking more closely at the issues and
open questions that need further studies. There are several such problems, for which more efforts and
new ideas are needed. Starting with the case of the critical Pomeron \( (\alpha_0 = \alpha_{oc}) \) one would like to know
the answer to the following questions:

1) Is the critical Pomeron relevant to phenomenology of high energy scattering? – We have seen in
section 2.5 that the present experimental value for \( \alpha_0 - 1 = \mu_0 \) is slightly larger than what is
theoretically calculated for \( \alpha_{oc} - 1 = \mu_{oc} \), but the uncertainties discussed there concerning both
numbers prevent us from drawing a definite conclusion.

2) If indeed the critical theory is the relevant one, what is the transition energy between the
perturbative description of high energy scattering and the scaling region? – The approach to scaling
has to be further studied and efforts have to be made in particular in the phenomenological analysis of
this issue. The detailed questions which have to be answered were discussed in sections 2.4 and 2.5.

3) The question of the accurate numerical values for the critical exponents and scaling functions is
of particular importance if we intend to increase the predictive power of the theory. Although the
different calculations of the critical exponents in table 1 seem to accumulate around certain values,
there is still a need for more reliable calculations.* As seen in section 7.3 the recent developments in
formulating the quantum spin analog model are very promising. This will be important if indeed the
critical theory is relevant to high energy scattering phenomenology.

Let us proceed now and discuss the ordered phase \( (\alpha_0 > \alpha_{oc}) \), where several important theoretical
issues relating to the structure of RFT in this subcritical region still exist and need further
examination. These open questions are:

4) What is the asymptotic behavior of the total cross section at \( D = 2 \)? Though an argument was
brought for a log\(^2\) \( s \) asymptotic behavior this has never been proven and there are several indications
suggesting that it might be different. A detailed analysis at \( D = 2 \) in the quantum spin model is
certainly needed.

5) The long standing problems which were solved in the case of \( \alpha_0 = \alpha_{oc} \) have to be studied again
for \( \alpha_0 > \alpha_{oc} \). Indeed, as seen in section 7.2, at \( D = 1 \) the Finkelstein–Kajantie problem does not exist
at \( \alpha_0 > \alpha_{oc} \), but other past inconsistencies were not discussed at the quantum level. This opens a whole
new set of problems to be solved.

6) It may be very useful, in order to study \( s \)-channel issues in the ordered phase, to formulate cut
Reggeon field theory on the lattice. This has not been done yet and in fact, even the classical solution
of CRFT was studied only for a very special case.*

7) The special role played by the classical solutions needs further studies in particular in regard to
the structure of the vacua of the theory.

8) The apparent conflicts with \( t \)-channel unitarity in the resultant expanding box discussed in
section 8 have to be understood. The analysis of the contribution of multi-box states may clarify the
source of this serious difficulty.

Finally, at a much more fundamental level lies the issue of understanding the construction of
the bare Pomeron from the underlying strong interaction theory. The progress made in this field is of
course interesting for its own merits and its relation to RFT was studied by several authors. The
interesting approach in refs. [67, 162], identifying the bare Pomeron with classes of Feynman graphs,

*Very recent new developments on these topics are presented in the Note added in proof.
sharing certain topological structure, does not rely on specific underlying strong interaction theory, though one has in mind a possible non-Abelian gauge theory. The expansion in terms of certain topologies (number of "handles") is equivalent to an expansion in $N^{-1}$ in the theory with $U(N)$ gauge symmetry. In certain phase space regions the topological expansion can be put in correspondence with RFT diagrams and in fact the topological expansion of $s$-channel discontinuities resulted in the same cut Reggeon field theory discussed in section 3.4, obeying the AGK cutting rules. Further studies along these lines will certainly be helpful in revealing the nature of the bare Pomeron.

The study of the nature of the $J$-plane singularities was carried out also by analyzing specific field theories. The high energy behavior of $\phi^3$ theory in six dimensions, which is exactly renormalizable, was analyzed in [60], where it was shown that when the theory is asymptotic free, moving Regge poles dominate the small $t$ region. If, however, the deep inelastic limit is controlled by a non-zero fixed point (which implies $q^2$ power corrections to Bjorken scaling), then the $t = 0$ region is dominated by fixed cuts. The complex angular momentum structure of meson scattering amplitudes was also considered in quantum chromodynamics (QCD) in two dimensions in the first order in $N^{-1}$ [45, 82]. To that order the theory does not possess a bare Pomeron singularity, but non-diffractive scattering amplitudes show a typical Regge behavior $s^\gamma$. Another origin of the bare Pomeron was considered in [10].

If indeed the Pomeron intercept satisfies $\alpha_0 = \alpha_{0c}$, we will have to understand within the underlying theory the reason for this special relation among the bare parameters. Alternatively, we may find that $\alpha_0$ is indeed greater than $\alpha_{0c}$. In any case, studying RFT's phase directly in the bare theory is certainly interesting. Further studies will hopefully reveal the nature of the bare Pomeron. (More recent studies of the high energy behavior of non-abelian gauge theories are presented in the Note added in proof [2P] [3P] and [4P].)

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Note added in proof

Since this review had been prepared, many new papers related to the topics covered above have appeared. In this short Note added in proof several of these developments and their relation to the early works described in sections 2–8 will be briefly mentioned and thus refer the reader to the original works.

The importance of investigating the theory at $D = 0$ transverse dimensions was emphasized in section 4. At $D = 0$ RFT reduces to a quantum mechanical problem and provides an excellent laboratory for understanding the non-perturbative features of RFT at $\alpha > 1$. The activity in this direction was carried out by several groups. Ciafaloni, Le Bellac and Rossi [7P] have used the equivalence of the eigenvalue problem at $D = 0$ with a Schrödinger-type equation to confirm most of the previous results within a firm mathematical framework. At fixed $Y$ and $r_0$ the $S$ matrix in the quantum theory has a spectral representation analytic in $\Delta_0$ for all $-\infty < \Delta_0 = 1 - \alpha_0 < +\infty$. At $\Delta_0 < 0$ and $r_0/\Delta_0$ small (classical limit) there is a singularity in the classical action at $Y = Y_c$; perturbation theory still holds at $Y \leq Y_c$ but breaks
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It was shown that adding a four Pomeron coupling \( \lambda_0 \) does not affect these results as long as its strength is below the "magic" value \( r_0^2/\Delta_0 \) discussed in section 8 [36.1]. For \( \Delta_0 = 0 \) the Green's functions at fixed \( Y \) are the Borel sum of perturbation theory. (At fixed \( E \), perturbation theory is Borel summable only at \( \Delta_0 > 0 \) – see section 4 and ref. [36].) This work confirms also the results of [15] [20] concerning the existence of a state almost degenerate with the vacuum whose energy \( \sim \exp(\frac{-2\Delta_0^2}{r_0^2}) \) sets the scale for the high \( s \) behavior of the amplitude, and the existence of this tunneling effect in the formulation of ref. [7P] is essential for the analyticity in \( \Delta_0 \) at fixed \( Y \).

These results as well as early results in \( D = 0 \) [36] [119] were disputed by White [15P] and for a different reason, by Raikin and Ryskin [13P]. White [15P] had suggested to use the path integral formulation of RFT in order to distinguish between the present alternative formulations of the theory in [15] [20] [21] (section 7) versus [169] (section 8). A full agreement exists only at \( \Delta_0 = 0 \); at \( \Delta_0 > 0 \) the two formulations differ in principle and the final results are slightly different; however, the two alternatives give completely different results at \( \Delta_0 < 0 \). In the path integral formulation this is viewed as a different choice of path in the \( \psi \phi \) plane. The three quasi-stable fixed points in fig. 36, namely the origin and the points A and B, have classical solutions connecting them (instantons) and there is an essential difference between formulating the theory with contributions from the multi-instanton path \( \rightarrow B \rightarrow (0, 0) \rightarrow A \) and a theory formulated with the single instanton path \( A \rightarrow B \). The inclusion of the multi-instanton path gives rise to the tunneling effect when \( \Delta_0 < 0 \) as in refs. [15] [20], which finally gives rise to the expanding disc [21]. The alternative suggested by White [169] does not include this path and results in \( \alpha_R \sim 2 - \alpha_0 \) as \( \alpha_0 \rightarrow 1 \), and in general, \( \alpha_R < 1 \) for \( \alpha_0 > \alpha_{oc} \). (At \( \Delta_0 > 0 \) the inclusion of the multi-instanton path contributes only negligible non-perturbative terms.) There are several important points emphasized by White [15P] that makes his choice of path very appealing: (1) Since tunneling does not exist at \( D \neq 0 \), one way to define the \( D = 0 \) theory as the \( \alpha' \rightarrow 0 \) limit of the \( D = 2 \) theory would be to avoid the inclusion of the multi-instanton path at \( D = 0 \). (2) The unwanted vacuum production of Pomerons which is absent in perturbation theory can be avoided by formulating the theory in such a way that the perturbative vacuum decouples, and, indeed, avoiding the multi-instanton path will do exactly that. Unfortunately, the path integral formulation is not rigorous and further work is needed, however, since the known results are essentially reproduced in this scheme for the case \( \Delta_0 \geq 0 \), it is hoped that its extension to \( \Delta_0 < 0 \) is correct, though it has not been proven yet.

Tan [14] studied the effect of quantum stability conditions on the \( D = 0 \) theory and found that the classical permutation symmetry between the three stationary points in fig. 36(A, B and \( (0, 0) \)) is broken when one requires these conditions; it then follows that tunneling at \( D = 0 \) and \( \Delta_0 < 0 \) is excluded. The mass spectrum is however not changed and the state with energy \( \sim \exp\left(\frac{-2\Delta_0^2}{r_0^2}\right) \) appears because of a certain mixture of classical and quantum configuration. The bare vacuum is nearly a pure instanton at \( \Delta_0 < 0 \) and the results are maintained if one adds a "magic" quartic Pomeron interaction \( \lambda_0 = \lambda_m = -r_0^2/\Delta_0 \); tunneling reappears in a limited range for \( \lambda_0 \) away from \( \lambda_m \).

As pointed out in section 7 (see below eq. (7.33)), away from the critical point the spin model encounters difficulties in the limit in which the lattice spacing tends to zero. These difficulties were removed in the ordered phase by adding a "magic" four Pomeron coupling [36.1]. The source of these difficulties is in the fact that the quantum spin model in section 7 neglects degrees of freedom of the system. Though these degrees are not important at \( \alpha_0 = \alpha_{oc} \), they do have an effect away from \( \alpha_{oc} \). Alessandrini, Amati and Ciafaloni [1P] had applied the method of kink quantization to RFT in order to obtain a scheme that includes the effects of the degrees of freedom neglected in the spin model. The expanding disc behavior obtained in the spin model appears here at \( \alpha_0 > \alpha_{oc} \) as in their \( D = 1 \) calculation.

The effect of adding a general four Pomeron couplings to the RFT Hamiltonian was studied by Bronzan
who found that as $\alpha_0$ is increased, an infinite number of single site states pass through zero. This however does not cause a breakdown of the quantum spin model of sec. 7, though it employs only the first two lowest states. The breakdown is avoided if the lattice spacing is taken to zero since the intercept has to be changed by an amount proportional to $a^{-D}$ ($a$ – lattice spacing and $D$ is the transverse dimensions) in order that the next excited state will pass through zero. It was also found that the spin model cannot be generalized by adding a $\lambda_1\phi^3$ term in the Hamiltonian.

A very interesting activity in the direction of constructing Reggeon Field Theory (see also section 9 above) from a non-Abelian gauge theory was carried out by Bartels [2P], Bartels and Scanio [3P] and by Bronzan and Sugar [4P]. The study of high energy fixed momentum transfer of spontaneously broken non-Abelian gauge theories in ref. [4P] is based on existing leading logarithm calculations of Feynman diagrams up to the tenth order in perturbation theory in an SU(2) gauge theory. These results are described by Reggeon Field Theory whose Regge trajectories and coupling functions are calculated in the leading logs approximation. In the $I = 1$ channel, previous works had shown that the $j$ plane singularity can be interpreted as a Regge pole amplitude – the Reggeized vector meson. Bronzan and Sugar [4P] interpret the $I = 0, 2$ channels as Regge cut amplitude generated by two $I = 1$ Regge poles. This enables them to determine in their leading log calculation Reggeon coupling functions, their strength and transverse momentum dependence. Using an integral equation for the two Reggeon→two Reggeon kernel given previously by Lipatov, one finds a fixed cut at $j > 1$ in the $I = 0$ amplitude, though order by order in perturbation theory only moving cuts appear. It was argued by Lipatov that this fixed cut will disappear being replaced by a moving Regge pole whose intercept is however to the right of one, once asymptotic freedom is properly included. Using renormalization group methods similar to those of ref. [60] (see section 9) Bronzan and Sugar found that the fixed cut at $j = 1 + g^2(\ln 4)/\pi^2 > 1$ is not removed. The presence of appropriate Reggeon and Pomeron interactions generated by non leading logs is necessary in any case in order to restore $s$-channel unitarity and hence further work is needed in order to include these non-leading logs in the scheme.

A different scheme had been proposed by Bartels [2P]. Using a generalized Lipatov dispersion relation technique to calculate the high energy behavior of Feynman diagrams for a spontaneously broken SU(2) gauge theory, an attempt is made to formulate RFT with moving Regge poles in all $I = 0, 1, 2$ channels. When generalized to an SU($N > 2$) gauge theory, Bartels’ scheme shows that the appearance of bare Reggeons with $\Delta_0 < 1$ is inevitable and thus Reggeon interactions have to restore $s$-channel unitarity [3P]. The scheme prescribes the organization of the leading and non-leading logs into the necessary multi-Reggeon and Pomeron interactions.

One may hope to understand the occurrence of the phase transition at $\alpha_0 = \alpha_{0c}$ in the Pomeron channel by studying the dynamics and symmetry properties of these attempts to derive a Reggeon field theory. It is of course very important to understand which of the above results may still survive in the case of an unbroken symmetry with masless vector mesons. Furthermore, since all scattered states considered in these schemes are not supposed to appear as asymptotic states in a confining theory of strong interactions, many questions are left open concerning the effect of confinement on the above schemes.

In a recent work Cardy and Moshe [6P] had shown that multiple Pomeron couplings to external particles have an important role in determining the asymptotic behavior of the scattering amplitude. This study clarifies the relation of Reggeon calculus to a renormalizable field theory and investigates the renormalization scheme for composite operators in Reggeon Field Theory. The previous proposed renormalization procedure for the couplings of the Reggeon sector to particles, led to multiple counting problems and severe difficulties in the renormalization (see for example [7] and [12P]); these difficulties are not present in [6P].
As pointed out in section 9 above, the formulation of cut Reggeon Field Theory on a lattice would be very useful in order to study s-channel issues in the ordered phase. Marchesini and Trentadue [10P] had accomplished this task and investigated the inclusive distribution in the central region in the case $\alpha_0 > \alpha_{0c}$. The quantum spin model is defined in a similar way as the one in section 7 by calculating the single site spectrum and truncating all operators using the vacuum $\phi_0$, and the four states $\phi_i (i = 1-4)$ nearly degenerate with it. The functional relations between the various Green's functions discussed in section 3.4, serve here as a useful guideline and unsurprisingly also lead to results similar to those in section 7. The inclusive distribution is calculated and shown to follow a geometrical-expanding disc type pattern for $\alpha_0 \gg \alpha_{0c}$. Feynman scaling is violated by logarithms and inelastic unitarity is satisfied.

The work by Brower Furman and Moshe [43] reported in section 7.3 had been completed and resulted in very precise calculations of the critical exponents and the critical temperature. The calculations of the high temperature series were extended up to ten orders at $D = 2$ and twenty orders at $D = 1$ thus, fig. 46 has now another ten points between $1/9$ and $1/19$, which indeed fall on the straight line shown there. The use of Padé approximants enabled us to evaluate the critical temperatures and critical exponents very accurately ($T_c(D = 2) = 2.428 \pm 0.003$ and $T_c(D = 1) = 0.6063 \pm 0.00005$). The results for the critical exponents at $D = 2$ were added at the bottom of table 1 in section 1. (The exponents at $D = 1$ have even a smaller uncertainty, approximately one order of magnitude smaller than those at $D = 2$ [43].) Note that these values coincide with Cardy's calculations [63] in the continuous theory and thus the conjecture in section 7, that the Reggeon Quantum Spin Model belongs to the same universality class as the continuous RFT, is strongly supported. We also confirm several of the results obtained by renormalization group transformations [62] for the structure of the critical lines and the independence of the critical exponents on $(\rho_s/\rho_t)$ defined in eq. (7.39). The existence of precise predictions for the scaling behavior and the approach to scaling certainly encourages a phenomenological approach to the issue of the transition energy discussed in section 2.5 [11P].

Interesting theoretical and phenomenological work in introducing thresholds into RFT was carried out by Garcia et al. [8P] and by Masperi [9P].

References

[34] S. Brodsky, private communication.
For Note added in proof