Sporadic SICs and Exceptional Lie Algebras

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Abstract

Sometimes, mathematical oddities crowd in upon one another, and the exceptions to one classification scheme reveal themselves as fellow-travelers with the exceptions to a quite different taxonomy.

1 Preliminaries

A set of *equiangular lines* is a set of unit vectors in a d-dimensional vector space such that the magnitude of the inner product of any pair is constant:

$$|\langle v_j, v_k \rangle| = \begin{cases} 1, & j = k; \\ \alpha, & j \neq k. \end{cases} \tag{1}$$

The maximum number of equiangular lines in a space of dimension d (the so-called *Gerzon bound*) is $d(d+1)/2$ for real vector spaces and d^2 for complex. In the real case, the Gerzon bound is only known to be attained in dimensions 2, 3, 7 and 23, and we know it can't be attained in general. If you like peculiar alignments of mathematical topics, the appearance of 7 and 23 might make your ears prick up here. If you made the wild guess that the octonions and the Leech lattice are just around the corner... you'd be absolutely right. Meanwhile, the complex case is of interest for quantum information theory, because a set of d^2 equiangular lines in $\mathbb{C}^d$ specifies a *measurement* that can be performed upon a quantum-mechanical system. These measurements are highly symmetric, in that the lines which specify them are equiangular, and they are "informationally complete" in a sense that quantum theory makes precise. Thus, they are known as *SICs* [1, 2, 3, 4]. Unlike the real case, where we can only attain the Gerzon bound in a few sparse instances, it *appears* that a SIC exists for each dimension d, but nobody knows for sure yet.

Before SICs became a physics problem, constructions of d^2 complex equiangular lines were known for dimensions $d = 2$, 3 and 8. These arose from topics like higher-dimensional polytopes and generalizations thereof [5, 6, 7, 8]. Now, we have exact solutions for SICs in the following dimensions [9, 10]:

$$d = 2\text{–}21, 23, 24, 28, 30, 31, 35, 37, 39, 43, 48, 53, 120, 124, 195, 323. \tag{2}$$

Moreover, numerical solutions to high precision are known for the following cases:

$$d = 2\text{–}151, 168, 172, 199, 224, 228, 255, 259, 288, 292, 323, 327, 489, 528, 725, 844, 1155, 2208. \tag{3}$$

These lists have grown irregularly in the years since the quantum-information community first recognized the significance of SICs. (Andrew Scott's contributions deserve particular mention, as he has found many solutions by solo work [11] and in collaboration with Markus Grassl [3, 12].) It is fair to say that researchers feel that SICs *should* exist for all integers $d \geq 2$, but we have no proof one way or the other. The attempts to resolve this question have extended into algebraic number theory [10, 13, 14, 15, 16], an intensely theoretical avenue of research with the surprisingly practical application of converting numerical solutions into exact ones [17]. For additional (extensive) discussion, we refer to the review article [4] and the textbooks [18, 19].

In what follows, we will focus our attention mostly on the *sporadic SICs,* which comprise the SICs in dimensions 2 and 3, as well as one set of them in dimension 8 [20]. These SICs have been designated "sporadic" because they stand out in several ways, chiefly by residing outside the number-theoretic patterns observed for the rest of the known SICs [16]. After laying down some preliminaries, we will establish a connection between the sporadic SICs and the exceptional Lie algebras E_6, E_7 and E_8 by way of their root systems.

2 Quantum Measurements and Systems of Lines

A *positive-operator-valued measure* (POVM) is a set of "effects" (positive semidefinite operators satisfying $0 < E < I$) that furnish a resolution of the identity:

$$\sum_i E_i = \sum_i w_i \rho_i = I, \tag{4}$$

for some density operators $\{\rho_i\}$ and weights $\{w_i\}$. Note that taking the trace of both sides gives a normalization constraint for the weights in terms of the dimension of the Hilbert space. In this context, the Born Rule says that when we perform the measurement described by this POVM, we obtain the i-th outcome with probability

$$p(i) = \mathrm{tr}\,(\rho w_i \rho_i), \tag{5}$$

where ρ without a subscript denotes our quantum state for the system. The weighting w_i is, up to a constant, the probability we would assign to the i-th outcome if our state ρ were the maximally mixed state $\frac{1}{d}I$, the "state of maximal ignorance."

SICs are a special type of POVM. Given a set of d^2 equiangular unit vectors $\{|\pi_i\rangle\} \subset \mathbb{C}^d$, we can construct the operators which project onto them, and in turn we can rescale those projectors to form a set of effects:

$$E_i = \frac{1}{d}\Pi_i, \text{ where } \Pi_i = |\pi_i\rangle\langle\pi_i|. \tag{6}$$

The equiangularity condition on the $\{|\pi_i\rangle\}$ turns out to imply that the $\{\Pi_i\}$ are linearly independent, and thus they span the space of Hermitian operators on $\mathbb{C}^d$. Because the SIC projectors $\{\Pi_i\}$ form a basis for the space of Hermitian operators, we can express any quantum state ρ in terms of its (Hilbert–Schmidt) inner products with them. But, by the Born Rule, the inner product $\mathrm{tr}\,(\rho\Pi_i)$ is, apart from a factor $1/d$, just the probability of obtaining the i-th outcome of the SIC measurement $\{E_i\}$. The formula for reconstructing ρ given these probabilities is quite simple, thanks to the symmetry of the projectors:

$$\rho = \sum_i \left[(d+1)p(i) - \frac{1}{d}\right]\Pi_i, \tag{7}$$

where $p(i) = \mathrm{tr}\,(\rho E_i)$ by the Born Rule. This furnishes us with a map from quantum state space into the probability simplex, a map that is one-to-one but not onto. In other words, we can fix a SIC as a "reference measurement" and then transform between density matrices and probability distributions without ambiguity, but the set of *valid* probability distributions for our reference measurement is a proper subset of the probability simplex.

Because we can treat quantum states as probability distributions, we can apply the concepts and methods of probability theory to them, including Shannon's theory of information. The structures that I will discuss in the following sections came to my attention thanks to Shannon theory. In particular, the question of recurring interest is, "Out of all the extremal states of quantum state space — i.e., the 'pure' states $\rho = |\psi\rangle\langle\psi|$ — which *minimize* the Shannon entropy of their probabilistic representation?" I will focus on the cases of dimensions 2, 3 and 8, where the so-called sporadic SICs occur. In these cases, the information-theoretic question of minimizing Shannon entropy leads to intricate geometrical structures.

Any time we have a vector in $\mathbb{R}^3$ of length 1 or less, we can map it to a 2×2 Hermitian matrix by the formula

$$\rho = \frac{1}{2}\left(I + x\sigma_x + y\sigma_y + z\sigma_z\right), \tag{8}$$

where (x, y, z) are the Cartesian components of the vector and $(\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. This yields a positive semidefinite matrix ρ with trace equal to 1; when the vector has length 1, we have $\rho^2 = \rho$, and the density matrix is a rank-1 projector that can be written as $\rho = |\psi\rangle\langle\psi|$ for some vector $|\psi\rangle$.

Given any polyhedron of unit radius or less in $\mathbb{R}^3$, we can feed its vertices into the Bloch representation and obtain a set of density operators (which are pure states if they lie on the surface of the Bloch sphere). For a simple example, we can do a regular tetrahedron. Let s and s' take the values ± 1, and define

$$\rho_{s,s'} = \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(s\sigma_x + s'\sigma_y + ss'\sigma_z)\right). \tag{9}$$

To make these density matrices into a POVM, scale them down by the dimension. That is, take

$$E_{s,s'} = \frac{1}{2}\rho_{s,s'}. \tag{10}$$

Then, the four operators $E_{s,s'}$ will sum to the identity. In fact, they comprise a SIC.

By introducing a sign change, we can define another SIC,

$$\tilde{\rho}_{s,s'} = \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(s\sigma_x + s'\sigma_y - ss'\sigma_z)\right). \tag{11}$$

Each state in the original SIC is orthogonal to exactly one state in the second. In the Bloch sphere representation, orthogonal states correspond to *antipodal* points, so taking the four points that are antipodal to the vertices of our original tetrahedron forms a second tetrahedron. Together, the states of the two SICs form a cube inscribed in the Bloch sphere.

Here we have our first appearance of Shannon theory entering the story. With respect to the original SIC, the states $\{|\tilde{\pi}_i\rangle\}$ of the antipodal SIC all minimize the Shannon entropy. The two interlocking tetrahedra are, entropically speaking, dual structures.

3 E_6

In what follows, I will refer to H. S. M. Coxeter's *Regular Complex Polytopes* [7]. Coxeter devotes a goodly portion of chapter 12 to the *Hessian polyhedron*, which lives in $\mathbb{C}^3$ and has 27 vertices. These 27 vertices lie on nine diameters in sets of three apiece. He calls the polyhedron "Hessian" because its nine diameters and twelve planes of symmetry interlock in a particular way. Their incidences reproduce the *Hesse configuration*, a set of nine points on twelve lines such that four lines pass through each point and three points lie on each line. (This configuration is also known as the discrete affine plane on nine points, and as the Steiner triple system of order 3.)

Coxeter writes the 27 vertices of the Hessian polyhedron explicitly, in the following way. First, let ω be a cube root of unity, $\omega = e^{2\pi i/3}$. Then, construct the complex vectors

$$(0, \omega^\mu, -\omega^\nu),\ (-\omega^\nu, 0, \omega^\mu),\ (\omega^\mu, -\omega^\nu, 0), \tag{12}$$

where μ and ν range over the values 0, 1 and 2. As Coxeter notes, we could just as well let μ and ν range over 1, 2 and 3. He prefers this latter choice, because it invites a nice notation: We can write the vectors above as

$$0\mu\nu,\ \nu 0\mu,\ \mu\nu 0. \tag{13}$$

For example,

$$230 = (\omega^2, -1, 0), \tag{14}$$

and

$$103 = (-\omega, 0, 1). \tag{15}$$

Coxeter then points out that this notation was first introduced by Beniamino Segre, "as a notation for the 27 lines on a general cubic surface in complex projective 3-space. In that notation, two of the lines intersect if their symbols agree in just one place, but two of the lines are skew if their symbols agree in two places or nowhere." Consequently, the 27 vertices of the Hessian polyhedron correspond to the 27 lines on a cubic surface "in such a way that two of the lines are intersecting or skew according as the corresponding vertices are non-adjacent or adjacent."

Casting the Hessian polyhedron into the real space $\mathbb{R}^6$, we obtain the polytope known as 2_{21}, which is related to E_6, since the Coxeter group of 2_{21} is the Weyl group of E_6. The Weyl group of E_6 can also be thought of as the Galois group of the 27 lines on a cubic surface.

We make the connection to symmetric quantum measurements by following the trick that Coxeter uses in his Eq. (12.39). We transition from the space $\mathbb{C}^3$ to the complex projective plane by collecting the 27 vertices into equivalence classes, which we can write in homogeneous coordinates as follows:

$$\begin{array}{ccc} (0,1,-1), & (-1,0,1), & (1,-1,0) \\ (0,1,-\omega), & (-\omega,0,1), & (1,-\omega,0) \\ (0,1,-\omega^2), & (-\omega^2,0,1), & (1,-\omega^2,0) \end{array} \tag{16}$$

Let u and v be any two of these vectors. We find that

$$|\langle u,u\rangle|^2 = 4 \tag{17}$$

when the vectors coincide, and

$$|\langle u,v\rangle|^2 = 1 \tag{18}$$

when u and v are distinct. We can normalize these vectors to be quantum states on a three-dimensional Hilbert space by dividing each vector by $\sqrt{2}$.

We have found a SIC for $d = 3$. When properly normalized, Coxeter's vectors furnish a set of $d^2 = 9$ pure quantum states, such that the magnitude squared of the inner product between any two distinct states is $1/(d+1) = 1/4$.

Every known SIC has a group covariance property. Talking in terms of projectors, a SIC is a set of d^2 rank-1 projectors $\{\Pi_j\}$ on a d-dimensional Hilbert space that satisfy the Hilbert–Schmidt inner product condition

$$\mathrm{tr}\,(\Pi_j \Pi_k) = \frac{d\delta_{jk}+1}{d+1}. \tag{19}$$

These form a POVM if we rescale them by $1/d$. In every known case, we can compute all the projectors $\{\Pi_j\}$ by starting with one projector, call it Π_0, and then taking the orbit of Π_0 under the action of some group. The projector Π_0 is known as the *fiducial state.* (I don't know who picked the word "fiducial"; I think it was something Carl Caves decided on, way back.)

In all known cases but one, the group is the *Weyl–Heisenberg group* in dimension d. To define this group, fix an orthonormal basis $\{|n\rangle\}$ and define the operators X and Z such that

$$X|n\rangle = |n+1\rangle, \tag{20}$$

interpreting addition modulo d, and

$$Z|n\rangle = e^{2\pi i n/d}|n\rangle. \tag{21}$$

The Weyl–Heisenberg displacement operators are

$$D_{l\alpha} = (-e^{i\pi/d})^{l\alpha} X^l Z^\alpha. \tag{22}$$

Because the product of two displacement operators is another displacement operator, up to a phase factor, we can make them into a group by inventing group elements that are displacement operators multiplied by phase factors. This group has Weyl's name attached to it, because he invented X and Z back in 1925, while trying to figure out what the analogue of the canonical commutation relation would be for quantum mechanics on finite-dimensional Hilbert spaces [21, 22, 23]. It is also called the *generalized Pauli group*, because X and Z generalize the Pauli matrices σ_x and σ_z to higher dimensions (at the expense of no longer being Hermitian).

To relate this with the Coxeter construction we discussed earlier, turn the first of Coxeter's vectors into a column vector:

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{23}$$

Apply the X operator twice in succession to get the other two vectors in Coxeter's table (converted to column-vector format). Then, apply Z twice in succession to recover the right-hand column of Coxeter's table. Finally, apply X to these vectors again to effect cyclic shifts and fill out the table. This set of nine states is known as the *Hesse SIC*.

I have elsewhere written at some length about the cat's cradle of vectors we encounter in dimension $d = 3$. First, there's the Hesse SIC. Like all informationally complete POVMs, it defines a probabilistic representation of quantum state space, in this case mapping from 3×3 density matrices to the probability simplex for 9-outcome experiments. As suggested earlier, we can look for the pure states whose probabilistic representations minimize the Shannon entropy. The result is a set of twelve states, which sort themselves into four orthonormal bases of three states apiece. What's more, these bases are *mutually unbiased*: The Hilbert–Schmidt inner product of a state from one basis with any state from another is always constant. In a sense, the Hesse SIC has a "dual" structure, and that dual is a set of Mutually Unbiased Bases (MUB). This duality relation is rather intricate: Each of the 9 SIC states is orthogonal to exactly 4 of the MUB states, and each of the MUB states is orthogonal to exactly 3 SIC states [24].

An easy way to remember these relationships is to consider the finite affine plane on nine points. Each of the points corresponds to a SIC vector, and each of the lines correponds to a MUB vector, with point-line incidence implying orthogonality. The four bases are the four ways of carving up the plane into parallel lines (horizontals, verticals, diagonals and other diagonals).

Considering all the lines in the original structure that are orthogonal to a given line in the dual yields a maximal set of real equiangular lines in one fewer dimensions. (Oddly, I noticed this happening up in dimension 8 before I thought to check in dimension 3 [25], but we'll get to that soon.) To visualize the step from $\mathbb{C}^3$ to $\mathbb{R}^2$, we can use the Bloch sphere representation for two-dimensional quantum state space. Pick a state in the dual structure, i.e., one of the twelve MUB vectors. All the SIC vectors that are orthogonal to it must crowd into a 2-dimensional subspace. In other words, they all fit into a qubit-sized state space, and we can draw them on the Bloch sphere. When we do so, they are coplanar and lie at equal intervals around a great circle, a configuration sometimes called a *trine* [26]. This configuration is a maximal equiangular set of lines in the plane $\mathbb{R}^2$.

What happens if, starting with the Hesse SIC, you instead consider all the lines in the dual structure that are orthogonal to a given vector in the original? This yields a SIC in dimension 2. I don't know where in the literature that is written, but if feels like something Coxeter would have known.

Another path from the sporadic SICs to E_6 starts with the qubit SICs, i.e., regular tetrahedra inscribed in the Bloch sphere. Shrinking a tetrahedron, pulling its vertices closer to the origin, yields a type of quantum measurement (sometimes designated a SIM [27]) that has more intrinsic noise. Apparently, E_6 is part of the story of what happens when the noise level becomes maximal and the four outcomes of the measurement merge into a single degenerate case. This corresponds to a singularity in the space of all rotated and scaled tetrahedra centered at the origin. Resolving this singularity turns out to involve the Dynkin diagram of E_6: We invent a smooth manifold that maps to the space of tetrahedra, by a mapping that is one-to-one and onto everywhere except the origin. The pre-image of the origin in this smooth manifold is a set of six spheres, and two spheres intersect if and only if the corresponding vertices in the Dynkin diagram are connected [28].

4 E_7

We said a moment ago that the Weyl–Heisenberg group was the group we use in call cases but one. Now, we take on that exception.

We saw how to generate the Hesse SIC by taking the orbit of a fiducial state under the action of the $d = 3$ Weyl–Heisenberg group. Next, we will do something similar in $d = 8$. We start by defining the two states

$$|\psi_0^{\pm}\rangle \propto (-1 \pm 2i, 1, 1, 1, 1, 1, 1, 1)^{\mathrm{T}}. \tag{24}$$

Here, we are taking the transpose to make our states column vectors, and we are leaving out the dull part, in which we normalize the states to satisfy

$$\langle\psi_0^{+}|\psi_0^{+}\rangle = \langle\psi_0^{-}|\psi_0^{-}\rangle = 1. \tag{25}$$

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1110111011100001111011101110000111101110111000010001000100011110
1101110111010010110111011101001011011101110100100010001000101101
1011101110110100101110111011010010111011101101000100010001001011
0111011101111000011101110111100001110111011110001000100010000111
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Table 1: Four of the states from the $\{\Pi_i^-\}$ Hoggar-type SIC, written in the probabilistic representation of three-qubit state space provided by the $\{\Pi_i^+\}$ SIC. Up to an overall normalization by 1/36, these states are all binary sequences, i.e., they are uniform over their supports.

First, we focus on $|\psi_0^+\rangle$. To create a SIC from the fiducial vector $|\psi_0^+\rangle$, we take the set of Pauli matrices, including the identity as an honorary member: $\{I, \sigma_x, \sigma_y, \sigma_z\}$. We turn this set of four elements into a set of sixty-four elements by taking all tensor products of three elements. This creates the Pauli operators on three qubits. By computing the orbit of $|\psi_0^+\rangle$ under multiplication (equivalently, the orbit of $\Pi_0^+ = |\psi_0^+\rangle\langle\psi_0^+|$ under conjugation), we find a set of 64 states that together form a SIC set.

The same construction works for the other choice of sign, $|\psi_0^-\rangle$, creating another SIC with the same symmetry group. We can call both of them *SICs of Hoggar type.* With respect to the probabilistic representation furnished by the Π_0^+ SIC, the states of the Π_0^- SIC minimize the Shannon entropy, and vice versa [25, 29].

Recall that when we invented SICs for a single qubit, they were tetrahedra in the Bloch ball, and we could fit together two tetrahedral SICs such that each vector in one SIC was orthogonal (in the Bloch picture, antipodal) to exactly one vector in the other. The two Hoggar-type SICs made from the fiducial states Π_0^+ and Π_0^- satisfy the grown-up version of this relation: Each state in one is orthogonal to *exactly twenty-eight states* of the other.

We can understand these orthogonalities as corresponding to the antisymmetric elements of the three-qubit Pauli group. It is simplest to see why when we look for those elements of the Π_0^- SIC that are orthogonal to the projector Π_0^+. These satisfy

$$\mathrm{tr}\,(\Pi_0^+ D \Pi_0^- D^\dagger) = 0 \tag{26}$$

for some operator D that is the tensor product of three Pauli matrices. For which such tensor-product operators will this expression vanish? Intuitively speaking, the product $\Pi_0^+\Pi_0^-$ is a symmetric matrix, so if we want the trace to vanish, we ought to try introducing an asymmetry, but if we introduce too much, it will cancel out, on the "minus times a minus is a plus" principle. Recall the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{27}$$

Note that of these three matrices, only σ_y is antisymmetric, and also note that we have

$$\sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y. \tag{28}$$

This much is familiar, though that minus sign gets around. For example, it is the fuel that makes the GHZ thought-experiment go [30, 31], because it means that

$$\sigma_x \otimes \sigma_x \otimes \sigma_x = -(\sigma_x \otimes \sigma_z \otimes \sigma_z)(\sigma_z \otimes \sigma_x \otimes \sigma_z)(\sigma_z \otimes \sigma_z \otimes \sigma_x). \tag{29}$$

Let's consider the finite-dimensional Hilbert space made by composing three qubits. This state space is eight-dimensional, and we build the *three-qubit Pauli group* by taking tensor products of the Pauli matrices, considering the 2×2 identity matrix to be the zeroth Pauli operator. There are 64 matrices in the three-qubit Pauli group, and we can label them by six bits. The notation

$$\begin{pmatrix} m_1 & m_3 & m_5 \\ m_2 & m_4 & m_6 \end{pmatrix} \tag{30}$$

means to take the tensor product

$$\sigma_x^{m_1}\sigma_z^{m_2} \otimes (-i)^{m_3 m_4}\sigma_x^{m_3}\sigma_z^{m_4} \otimes (-i)^{m_5 m_6}\sigma_x^{m_5}\sigma_z^{m_6}. \tag{31}$$

Now, we ask: Of these 64 matrices, how many are symmetric and how many are antisymmetric? We can only get antisymmetry from σ_y, and (speaking heuristically) if we include too much antisymmetry, it will cancel out. More carefully put: We need an odd number of factors of σ_y in the tensor product to have the result be an antisymmetric matrix. Otherwise, it will come out symmetric. Consider the case where the first factor in the triple tensor product is σ_y. Then we have $(4-1)^2 = 9$ possibilities for the other two slots. The same holds true if we put the σ_y in the second or the third position. Finally, $\sigma_y \otimes \sigma_y \otimes \sigma_y$ is antisymmetric, meaning that we have $9 \cdot 3 + 1 = 28$ antisymmetric matrices in the three-qubit Pauli group. In the notation established above, they are the elements for which

$$m_1 m_2 + m_3 m_4 + m_5 m_6 = 1 \mod 2. \tag{32}$$

Moreover, these 28 antisymmetric matrices correspond exactly to the 28 bitangents of a quartic curve, and to pairs of opposite vertices of the Gosset polytope 3_{21}. In order to make this connection, we need to dig into the octonions.

To recap: Each of the 64 vectors (or, equivalently, projectors) in the Hoggar SIC is naturally labeled by a displacement operator, which up to an overall phase is the tensor product of three Pauli operators. Recall that we can write the Pauli operator σ_y as the product of σ_x and σ_z, up to a phase. Therefore, we can label each Hoggar-SIC vector by a pair of binary strings, each three bits in length. The bits indicate the power to which we raise the σ_x and σ_z generators on the respective qubits. The pair $(010, 101)$, for example, means that on the three qubits, we act with σ_x on the second, and we act with σ_z on the first and third. Likewise, $(000, 111)$ stands for the displacement operator which has a factor of σ_z on each qubit and no factors of σ_x at all.

There is a natural mapping from pairs of this form to pairs of unit octonions. Simply turn each triplet of bits into an integer and pick the corresponding unit from the set $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, where each of the e_j square to -1.

We can choose the labeling of the unit imaginary octonions so that the following nice property holds. Up to a sign, the product of two imaginary unit octonions is a third, whose index is the XOR of the indices of the units being multiplied. For example, in binary, $1 = 001$ and $4 = 100$; the XOR of these is $101 = 5$, and e_1 times e_4 is e_5.

Translate the Cayley–Graves table here into binary if enlightenment has not yet struck:

$e_i e_j$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

(33)

So, each projector in the Hoggar SIC is labeled by a pair of octonions, and the group structure of the displacement operators is, almost, octonion multiplication. There are sign factors all over the place, but for this purpose, we can neglect them. They will crop up again soon, in a rather pretty way.

Another way to express the Cayley–Graves multiplication table is with the *Fano plane*, a set of seven points grouped into seven lines that has been called "the combinatorialist's coat of arms". We can label the seven points with the imaginary octonions e_1 through e_7. When drawn on the page, a useful presentation of the Fano plane has the point e_4 in the middle and, reading clockwise, the points e_1, e_7, e_2, e_5, e_3 and e_6 around it in a regular triangle. The three sides and three altitudes of this triangle, along with the inscribed circle, provide the seven lines: (e_1, e_2, e_3), (e_1, e_4, e_5), (e_1, e_7, e_6), (e_2, e_4, e_6), (e_2, e_5, e_7), (e_3, e_4, e_7), (e_3, e_6, e_5). It is apparent that each line contains three points, and it is easy to check that each point lies within three distinct lines, and that each pair of lines intersect at a single point. One consequence of this is that if we take the *incidence matrix* of the Fano plane, writing a 1 in the ij-th entry if line i contains point

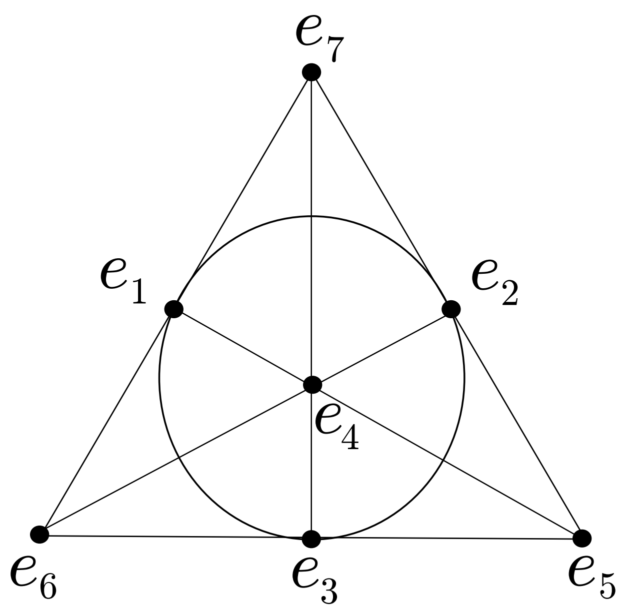

Figure 1: The Fano plane: a symmetrical arrangement of seven points and seven lines. Each point lies on three lines, each line contains three points, and every pair of lines intersect in a single point.

j, then every two rows of the matrix have exactly the same overlap:

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}. \quad (34)$$

The rows of the incidence matrix furnish us with seven equiangular lines in $\mathbb{R}^7$. We can build upon this by considering the *signs* in the Cayley–Graves multiplication table, which we can represent by adding *orientations* to the lines of the Fano plane. Start by taking the first row of the incidence matrix M, which corresponds to the line (e_1, e_2, e_3), and give it all possible choices of sign by multiplying by the elements not on that line. Multiplying by e_4, e_5, e_6 and e_7 respectively, we get

$$\begin{pmatrix} + & + & + & 0 & 0 & 0 & 0 \\ - & + & - & 0 & 0 & 0 & 0 \\ - & - & + & 0 & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (35)$$

The sign we record here is simply the sign we find in the corresponding entry of the Cayley–Graves table. Doing this with all seven lines of the Fano plane, we obtain a set of 28 vectors, each one given by a choice of a line and a point not on that line. Moreover, *all of these vectors are equiangular.* This is easily checked: For any two vectors derived from the same Fano line, two of the terms in the inner product will cancel, leaving an overlap of magnitude 1. And for any two vectors derived from different Fano lines, the overlap always has magnitude 1 because each pair of lines always meets at exactly one point. Van Lint and Seidel noted that the incidence matrix of the Fano plane could be augmented into a full set of 28 equiangular lines [32, 33], but to my knowledge, extracting the necessary choices of sign from octonion multiplication is not reported in the literature.

So, the 28 lines in a maximal equiangular set in $\mathbb{R}^7$ correspond to point-line pairs in the Fano plane, where the point and the line are *not* coincident. In discrete geometry, the combination of a line and a point not on that line is known as an *anti-flag.* It is straightforward to show from Eq. (32) that the antisymmetric matrices in the three-qubit Pauli group also correspond to the anti-flags of the Fano plane: Simply take the powers of the σ_x operators to specify a point and the powers of the σ_z operators to specify a line [25].

Two fun things have happened here: First, we started with complex equiangular lines. By carefully considering the orthogonalities between two sets of complex equiangular lines, we arrived at a maximal set of real equiangular lines in $\mathbb{R}^7$. And since one cannot actually fit more equiangular lines into $\mathbb{R}^8$ than into $\mathbb{R}^7$, we have a connection between a maximal set of equiangular lines in $\mathbb{C}^8$ and a maximal set of them in $\mathbb{R}^8$.

Second, our equiangular lines in $\mathbb{R}^7$ are the diameters of the Gosset polytope 3_{21}. And because we have made our way to the polytope 3_{21}, we have arrived at E_7.

5 E_8

To make this connection, we consider the stabilizer of the fiducial vector, i.e., the group of unitaries that map the SIC set to itself, leaving the fiducial where it is and permuting the other $d^2 - 1$ vectors. Huangjun Zhu observed that the stabilizer of any fiducial for a Hoggar-type SIC is isomorphic to the group of 3×3 unitary matrices over the finite field of order 9 [34, 35]. This group is sometimes written $U_3(3)$ or $\mathrm{PSU}(3,3)$. In turn, this group is up to a factor $\mathbb{Z}_2$ isomorphic to $G_2(2)$, the automorphism group of the Cayley integers, a subset of the octonions also known as the *octavians* [36]. Up to an overall scaling, the lattice of octavians is also the lattice known as E_8.

The octavian lattice contains a great deal of arithmetic structure. Of particular note is that it contains 240 elements of norm 1. In addition to the familiar $+1$ and -1, which have order 1 and 2 respectively, there are 56 units of order 3, 56 units of order 6 and 126 units of order 4. The odd-order units generate subrings of the octavians that are isomorphic to the *Eisenstein integers* and the *Hurwitz integers,* lattices in the complex numbers and the quaternions [36]. From the symmetries of these lattices, we can in fact read off the stabilizer groups for fiducials of the qubit and Hesse SICs [20]. It is as if the sporadic SICs are drawing their strength from the octonions.

Before moving on, we pause to note how peculiar it is that by trying to find a nice packing of complex unit vectors, we ended up talking about an optimal packing of Euclidean hyperspheres [37].

6 The Hesse SIC and the Regular Icosahedron

In the previous sections, we uncovered correspondences between equiangular lines in $\mathbb{C}^3$ and $\mathbb{R}^2$, and between $\mathbb{C}^8$ and $\mathbb{R}^7$. It would be nice to have a connection between $\mathbb{C}^4$ and $\mathbb{R}^3$, but I have not found one yet. Instead, there is a slightly different relationship that brings $\mathbb{R}^3$ into the picture.

Suppose that, unaccountably, we wished to build the Hesse SIC, but in *real* vector space. What might this even mean? It would entail finding a fiducial vector and an appropriate group, closely analogous to the qutrit Weyl–Heisenberg group, such that the orbit of said fiducial is a maximal set of equiangular lines. How big would such a set of lines be? Recall that the Gerzon bound is d^2 for $\mathbb{C}^d$, but only $d(d+1)/2$ in $\mathbb{R}^d$. In both cases, this is essentially because those values are the dimensions of the appropriate operator spaces. It is not difficult to show that, if the Gerzon bound is attained, the magnitude of the inner product between the vectors is $1/\sqrt{d+1}$ in $\mathbb{C}^d$ and $1/\sqrt{d+2}$ in $\mathbb{R}^d$.

We are familiar with the complex case, in which we define a shift operator X and a phase operator Z that both have order d. A cyclic shift is nice and simple, so we'd like to keep that idea, but the only "phase" we have to work with is the choice of positive or negative sign. So, let us consider the operators

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

The shift operator X still satisfies $X^3 = I$, while for the phase operator Z, we now have $Z^2 = I$.

What group can we make from these operators? Note that

$$(ZX)^3 = -I, \tag{37}$$

and so

$$(-Z)^2 = X^3 = (-ZX)^3 = I, \tag{38}$$

meaning that the operators X and $-Z$ generate the *tetrahedral group,* so designated because it is isomorphic to the rotational symmetry group of a regular tetrahedron. Equivalently, we can use Z as a generator, since $-Z = (ZX)^3 Z$ by the above.

Now, we want to take the orbit of a vector under this group! But what vector? It should not be an eigenvector of X or of Z, for then we know we could never get a full set. Therefore, we don't want a flat vector, nor do we want any of the basis vectors, so we go for the next simplest thing:

$$v = \begin{pmatrix} 0 \\ 1 \\ y \end{pmatrix}, \tag{39}$$

where y is a real number. We now have

$$Zv = \begin{pmatrix} 0 \\ -1 \\ y \end{pmatrix} \tag{40}$$

and also

$$X^2 v = \begin{pmatrix} 1 \\ y \\ 0 \end{pmatrix}, \tag{41}$$

so if we want equality between the inner products,

$$\langle Zv, v \rangle = \langle X^2 v, v \rangle, \tag{42}$$

then we need to have

$$-1 + y^2 = y. \tag{43}$$

The positive solution to this quadratic equation is

$$y = \frac{1+\sqrt{5}}{2}, \tag{44}$$

so we can in fact take our y to be ϕ, the golden ratio.

In the group we defined above, X performs cyclic shifts, Z changes the relative phase of the components, and we have the freedom to flip all the signs. Therefore, the orbit of the fiducial v is the set of twelve vectors

$$\begin{pmatrix} 0 \\ \pm 1 \\ \pm\phi \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \pm\phi \\ 0 \end{pmatrix}, \begin{pmatrix} \pm\phi \\ 0 \\ \pm 1 \end{pmatrix}. \tag{45}$$

These are the vertices of a regular icosahedron, and the diagonals of that icosahedron are six equiangular lines. The inner products between these vectors are always $\pm\phi$. Since $d(d+1)/2 = 6$, there cannot be any larger set of equiangular lines in $\mathbb{R}^3$.

Recall that the reciprocal of the golden ratio ϕ is

$$\phi^{-1} = \frac{2}{1+\sqrt{5}} \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{2-2\sqrt{5}}{1-5} = \frac{-1+\sqrt{5}}{2} = \phi - 1. \tag{46}$$

The golden ratio ϕ is a root of the monic polynomial $y^2 - y - 1$, and being a root of a monic polynomial with integer coefficients, it is consequently an algebraic integer. The same holds for its reciprocal, so ϕ^{-1} is also an algebraic integer, making the two of them *units* in the number field $\mathbb{Q}(\sqrt{5})$.

To summarize: For the diagonals of the regular icosahedron, the vector components are given by the units of the "golden field" $\mathbb{Q}(\sqrt{5})$. But it has been discovered [15] that the vector components for the Weyl–Heisenberg SICs in dimension *four* are derived from a unit in the *ray class field over* $\mathbb{Q}(\sqrt{5})$. Therefore, the icosahedron is the Euclidean version of the Hesse SIC, and the SICs in $d = 4$ are the number-theoretic extension of the icosahedron.

Futhermore, it has been observed empirically that in dimensions

$$d_k = \phi^{2k} + \phi^{-2k} + 1, \tag{47}$$

there exist Weyl–Heisenberg SICs with additional group-theoretic properties that make their exact expressions easier to find. These are known as *Fibonacci–Lucas SICs* [12].

7 Speculations Concering Finite Simple Groups

There are exactly four known cases where the Gerzon bound can be attained in $\mathbb{R}^d$: when $d = 2$, 3, 7 and 23. Three out of these four examples relate to SICs, specifically to the sporadic SICs. We can obtain the maximal equiangular sets in $\mathbb{R}^2$ and $\mathbb{R}^7$ from SICs in $\mathbb{C}^3$ and $\mathbb{C}^8$ respectively, while the set in $\mathbb{R}^3$ turns out to be the real analogue of our example in $\mathbb{C}^3$. All of this raises a natural question: What about $\mathbb{R}^{23}$? Does the equiangular set there descend from a SIC in $\mathbb{C}^{24}$? That, nobody knows.

We do know that the maximal equiangular line set in $\mathbb{R}^{23}$ can be extracted from the Leech lattice [38]. It contains 276 lines, and its automorphism group is Conway's group Co_3 [39].

We now recall that the stabilizer subgroup for each vector in a Hoggar-type SIC is isomorphic to the projective special unitary group of 3×3 matrices over the finite field of order 9, known for short as $\mathrm{PSU}(3, 3)$. Finite-group theorists also refer to this structure as $\mathrm{U}_3(3)$, and as $\mathrm{G}_2(2)'$, since it is isomorphic to the commutator subgroup of the automorphism group of the *octavians,* the integer octonions. The symbol $\mathrm{G}_2(2)'$ arises because the automorphism group of the octonions is called G_2, when we focus on the octavians we add a 2 in parentheses, and when we form the subgroup of all the commutators, we affix a prime.

We also recall that, given one SIC of Hoggar type, we can construct another by antiunitary conjugation, and each vector in the first SIC will be orthogonal to exactly 28 vectors out of the 64 in the other SIC. Furthermore, the Hilbert–Schmidt inner products that are not zero are all equal. Said another way, if we use the first SIC to define a probabilistic representation of three-qubit state space, then each vector in the second SIC is a probability distribution that is uniform across its support. Up to normalization, such a probability distribution is a binary string composed of 28 zeros and 36 ones.

Let $\{\Pi_j^+\}$ be a SIC of Hoggar type, and let $\{\Pi_j^-\}$ be its conjugate SIC. Suppose that U is a unitary that permutes the $\{\Pi_j^+\}$. Then a linear combination of 36 equally weighted projectors drawn from $\{\Pi_j^+\}$ will be sent to a linear combination of 36 equally weighted projectors from the set $\{\Pi_j^+\}$, possibly a different combination. But the only sequences of 36 ones and 28 zeros that correspond to valid quantum states are the representations of $\{\Pi_j^-\}$. Therefore, a unitary that shuffles the + SIC will also shuffle the − SIC. Furthermore, a unitary that *stabilizes* a projector, say Π_0^-, must permute the $\{\Pi_j^+\}$ in such a way that 1's go to 1's and 0's go to 0's.

To repeat: Because each SIC provides a basis, we can uniquely specify a vector in one SIC by listing the vectors in the other SIC with which it has nonzero overlap.

A unitary symmetry of one SIC set corresponds to a permutation of the other. Using the second SIC to define a representation of the state space, each vector in the first SIC is essentially a binary string, and sending one vector to another permutes the 1's and 0's. In particular, a unitary that stabilizes a vector in one SIC must permute the vectors of the second SIC in such a way that the list of 1's and 0's remains the same. The 1's can be permuted among themselves, and so can the 0's, but the binary sequence as a whole does not change.

It would be nice to have a way of visualizing this with a more tangible structure than eight-dimensional complex Hilbert space — something like a graph. Thinking about the permutations of 36 vectors, we imagine a graph on 36 vertices, and we try to draw it in such a way that its group of symmetries is isomorphic to the stabilizer group of a Hoggar fiducial. Can this be done? Well, almost — that is, up to a "factor of two":

http://www.win.tue.nl/~aeb/drg/graphs/U3_3.html

Now, is there a way to illustrate the structure of both SICs together as a graph? We want to record the fact that each vector in one SIC is nonorthogonal to exactly 36 vectors of the other, that the stabilizer of each vector is $\mathrm{PSU}(3,3)$, and that a stabilizer unitary shuffles the nonorthogonal set within itself. So, we start with one vertex to represent a fiducial vector, then we add 63 more vertices to stand for the other vectors in the first SIC, and then we add 36 vertices to represent the vectors in the second SIC that are nonorthogonal to the fiducial of the first. We'd like to connect the vertices in such a way that the stabilizer of any vertex is isomorphic to $\mathrm{PSU}(3,3)$. In fact, because any vector in either SIC can be identified by the list of the 36 nonorthogonal vectors in the other, the graph should look locally like the $\mathrm{U}_3(3)$ graph everywhere! Is this possible?

http://www.win.tue.nl/~aeb/drg/graphs/HallJanko.html

And the automorphism group of this graph is, up to a pesky "factor of 2", the Hall–Janko finite simple group. To say it more neatly, the Hall–Janko group is an index-2 subgroup of the graph's automorphism group.

That's what we get when we think about the permutations of the 1's. What about $\mathrm{PSU}(3,3)$ acting to permute the 0's?

This seems to lead us in the direction of the *Rudvalis group.* Wilson's textbook *The Finite Simple Groups* has this to say (§5.9.3):

> The Rudvalis group has order 145 926 144 000 $= 2^{14}.3^3.5^3.7.13.29$, and its smallest representations have degree 28. These are actually representations of the double cover $2 \cdot \mathrm{Ru}$ over the complex numbers, or over any field of odd characteristic containing a square root of -1, but they also give rise to representations of the simple group Ru over the field $\mathbb{F}_2$ of order 2.

Wilson then describes in some detail the 28-dimensional complex representations of $2 \cdot \mathrm{Ru}$, using a basis in which $2^6 \cdot \mathrm{G}_2(2)$ appears as the monomial subgroup [40].

Ru has a maximal subgroup given by a semidirect product $2^6 : \mathrm{G}_2(2)' : 2$. This is what first caught my eye. Neglecting the issue of the group actions required to define the semidirect products, consider the factors: we have $\mathrm{G}_2(2)'$ from the Hoggar stabilizer, 2 from conjugation and 2^6 from the three-qubit Pauli group.

This is reminiscent of a theorem proved by O'Nan [41]:

> Let G be a finite simple group having an elementary abelian subgroup E of order 64 such that E is a Sylow 2-subgroup of the centralizer of E in G and the quotient of the normalizer of E in G by the centralizer is isomorphic to the group $\mathrm{G}_2(2)$ or its commutator subgroup $\mathrm{G}_2(2)'$. Then G is isomorphic to the Rudvalis group.

The peculiar thing is that the Hall–Janko group is part of the "Happy Family", i.e., it is a subgroup of the Monster, while the Rudvalis group is a "pariah", floating off to the side. The Hoggar SIC almost seems to be acting as an intermediary between the two finite simple groups, one of which fits within the Monster while the other does not.

Finally, what connects the largest of sporadic simple groups with the second-smallest among quantum systems?

I was greatly amused to find the finite affine plane on nine points also appearing in the theory of the Monster group and the Moonshine module [42, 43]. In that case, the 9 points and 12 lines correspond to involution automorphisms. All the point-involutions commute with one another, and all the line-involutions commute with each other as well. The order of the product of a line-involution and a point-involution depends on whether the line and the point are incident or not.

This is probably of no great consequence — just an accident of the same small structures appearing in different places, because there are only so many small structures to go around. But it's a cute accident all the same.

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